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Mumford-Tate groups and Hodge classes on Abelian varieties of low dimension

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Introduction

The Hodge conjecture on the algebraicity of cohomology classes is probably one of the most well-known open problems in modern mathematics, and certainly a topic of central interest in analytic and algebraic geometry.

Despite the remarkable number of contributions, however, the conjecture still seems to be far from a complete solution, even for Abelian varieties, which are relatively well understood.

In this thesis we focus on the Hodge conjecture and related questions in the setting of Abelian varieties, which, while exhibiting most of the richness of the general problem, allows the development of specific tools that have led to important partial results. In this context it is rather natural to introduce the notion of an abstract Hodge structure, a rational vector space equipped with some extra structure at the level of $\mathbb{C}$-points.

In turn, to every Hodge structure we can associate a certain algebraic group, called its Mumford-Tate group, which allows a purely representation-theoretic description of Hodge classes. In principle, once the Mumford-Tate group of a Hodge structure is known, the computation of Hodge classes is reduced to a problem in invariant theory, and this is often enough to determine the whole Hodge ring, sometimes not just for the Abelian variety we started with but for its powers as well.

On the other hand, when the Abelian variety A is defined over a number field K, another long-standing conjecture, formulated by Tate, makes predictions about the algebraicity of certain (étale) cohomology classes.

The striking similarities between these two seemingly unrelated statements led Mumford, Tate and Serre to conjecture that a close connection should exist between the Mumford-Tate group of A and the action of the absolute Galois group of K on the ℓ-adic Tate module $T_\ell(A)$. This statement is now known as the Mumford-Tate conjecture and has an important part to play in this work.

A large part of this thesis is based on the paper "Hodge classes and Tate classes on simple abelian fourfolds" ([MZ95]), where a criterion is given for the existence, on simple Abelian varieties of dimension 4, of "exceptional" classes, namely classes that we expect to be algebraic but do not lie in the algebra generated by divisor classes.

The tools developed along the way actually allow the analysis of many other cases, and following Ribet, Serre, Tanke'ev and many others we recount the proofs of the Hodge and Tate conjectures for Abelian varieties satisfying various combinations of additional requirements on the dimension and on the endomorphism algebra.

After the truth of the so-called minuscule weights conjecture was established by Pink in [Pin98], it has become possible to give unified proofs that work equally well in the complex and ℓ-adic case; also, in many circumstances the representation-theoretic properties of the Mumford-Tate group allow its precise determination, and the ℓ-adic counterparts of the same arguments are enough to prove the Mumford-Tate conjecture by computing both sides of the predicted equality. Whenever possible, we try to adopt this kind of approach, in order to emphasize the similarities between the two conjectures.

We conclude this introduction with a brief outline of the material to follow.

In the first two chapters we discuss various preliminaries, especially regarding reductive algebraic groups and their representation theory; as most of the material is fairly standard, proofs are kept to the bare minimum. Chapter 3 then introduces the Mumford-Tate and Hodge groups, our main objects of interest, and contains a list of their most important properties. We also prove a characterization of the Mumford-Tate group (Prop. 3.1.2) which is often stated without proof in the literature, and will play a relevant part in subsequent sections.

The following chapter consists entirely of an account of what Mumford-Tate groups can look like in low dimension, namely for varieties of dimension 1 or 2. This is essentially an elaboration of Examples 5.4 and 5.7 and Exercise 5.6 of [Mooa], presenting the arguments from the reference in greater detail.

In Chapter 5 the precise statements are given for the Hodge, Tate and Mumford-Tate conjectures, along with some background and a series of results that will allow us to take a unified approach to both geometric and ℓ-adic questions.

Chapters 6, 7 and 8 are the core of this work; the first of these is dedicated to the computation of Mumford-Tate groups for some Abelian varieties with additional restrictions on both the endomorphism algebra and the dimension, whilst in the other two we drop any assumption on the endomorphism algebra. We are then able to obtain complete results for varieties of prime dimension; the next interesting case - namely, varieties of dimension 4 - is a difficult problem in its own right: it shall be the subject of the last Chapter, where we give a complete account of the main theorem of Moonen and Zarhin, again presenting detailed proofs, expanding computations and expliciting the full argument for varieties of type $\mathrm{IV}(2,1)$, a part of which was explained only briefly in [MZ95].

CHAPTER 1

Preliminaries

We collect here some useful results and definitions, starting with the notion that underlies everything we deal with in this work:

Definition 1.0.1. Let V be a (finite dimensional) $\mathbb{Q}$-vector space. On the complex vector space $V \otimes_{\mathbb{Q}} \mathbb{C}$ we have a notion of complex conjugation, induced by $\overline{v \otimes z} = v \otimes \bar{z}$ for all $v \in V$ and $z \in \mathbb{C}$. A $\mathbb{Q}$-**Hodge structure** of weight $m \in \mathbb{Z}$ is the data of V together with a decomposition

$$V \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{p+q=m} V^{p,q},$$

where each $V^{p,q}$ is a complex vector space and $\overline{V^{p,q}} = V^{q,p}$.

The **type** of a Hodge structure is the collection of the pairs (p, q) such that $V^{p,q}$ is non-trivial, and a vector $v \in V$ is called a **Hodge class** if it belongs to $V^{0,0}$.

1.1 Algebraic groups

Throughout, let k be a field and let $\bar{k}$ denote a fixed separable closure of k.

As it is well known, $\mathbb{G}_{m,k} := \operatorname{Spec} k\left[X, \frac{1}{X}\right]$ can be endowed with the structure of an affine algebraic group over k, called the **multiplicative group** of k. A **split torus** over k is an affine algebraic group of the form $\mathbb{G}_{m,k}^n$ for a certain (positive) integer n.

More generally, an **algebraic torus** over k is an affine algebraic group G over k such that $G_{\bar{k}}$ is a split torus.

1.1.1 Characters and cocharacters

Let T be a torus over k. Define the **character group** (resp. **cocharacter group**) of T to be

$$X^*(T) = \operatorname{Hom}\left(T_{\overline{k}}, \mathbb{G}_{m,\overline{k}}\right) \quad \left(\text{resp. } X_*(T) = \operatorname{Hom}\left(\mathbb{G}_{m,\overline{k}}, T_{\overline{k}}\right)\right).$$

The assignments $T \mapsto X^*(T)$ and $T \mapsto X_*(T)$ are clearly functorial in T, but in fact much more is true:

Theorem 1.1.1. *Let Γ be the absolute Galois group of k, i.e.* $\operatorname{Gal}\left(\overline{k}/k\right)$*.*

Then the functor X^* establishes a (contravariant) equivalence between the category of algebraic tori over k and the category of free abelian groups of finite rank equipped with a continuous action of Γ. Similarly, the functor X_* is an equivalence between the same two categories.

1.1.2 Weil restriction of scalars

Let K/k be a finite extension of fields and let G be an affine group over K, thought of as a functor

$$G : (\mathrm{Alg}_K) \to (\mathrm{Groups})$$

such that the underlying set-valued functor is representable. Then the functor

$$\begin{array}{rccc} \operatorname{Res}_{K/k} G : & (\mathrm{Alg}_k) & \to & (\mathrm{Groups}) \\ & A & \mapsto & G(A \otimes_k K) \end{array}$$

is an affine group over k in the above sense, called the **Weil restriction** of G to k.

Remark 1.1.2. By abstract nonsense (Yoneda lemma) the above property characterizes $\operatorname{Res}_{K/k} G$ up to unique isomorphism.

1.1.3 Lie algebras and the Weil restriction of scalars

Since we will almost always work with Lie algebras instead of their group counterparts, it is a natural question to ask what happens to them when applying the restriction of scalars functor. In this paragraph, following [BGK04], we give an answer to this question.

Let E/K be a separable field extension and Θ an algebraic group defined over E. Here we restrict ourselves to the case of affine algebraic groups, which is the only one we will need. There are two natural constructions that, given Θ, yield a Lie algebra over K:

- we can first consider the group $\Theta|_K := \operatorname{Res}_{E/K}(\Theta)$, which is by construction an algebraic group over K, and take its Lie algebra $\mathfrak{h} = \operatorname{Lie}(\Theta|_K)$;

- or we can take the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\Theta)$, which is a vector space over E, forget its structure as E-vector space and simply regard it as a K-vector space. This gives a K-Lie algebra (keeping the same bracket).

  In this case we will write $\mathrm{Res}_{E/K}(\mathfrak{g})$, $\mathrm{Res}_{E/K}$ being the forgetful functor $\mathrm{Lie}_E \to \mathrm{Lie}_K$.

The important fact about these construction is that they actually yield the same object:

Theorem 1.1.3. *With the above notation,* $\mathfrak{h} \cong \mathrm{Res}_{E/K}(\mathfrak{g})$

Definition 1.1.4. We will refer to $\mathrm{Res}_{E/K}(\mathfrak{g})$ as 'the Lie algebra of Θ regarded (as a Lie algebra) over K'.

Before proving the theorem we need to give a more explicit description of $\mathrm{Res}_{E/K}(\Theta)$.

Suppose $\Theta = \mathrm{Spec}(A)$, where

$$A \cong \frac{E[x_1,\dots,x_r]}{I}$$

is a finitely generated E-algebra, $I = (f_1(X),\dots,f_s(X))$ is an ideal of $E[X]$ and X denotes the multivariable $x_1,\dots,x_r$.

Fix a separable closure $\overline{K}$ of K and denote $\sigma_1,\dots,\sigma_n$ the different embeddings of E in $\overline{K}$; let furthermore M be the composite of $E_1 = \sigma_1(E),\dots,E_n = \sigma_n(E)$ inside $\overline{K}$. We can then define $A^{\sigma_i} := A \otimes_{E,\sigma_i} M$ and form the E-algebra

$$\overline{A} = A^{\sigma_1} \otimes_M A^{\sigma_2} \otimes_M \dots \otimes_M A^{\sigma_n}.$$

In order to fix notations, write $A^{\sigma_i} = M[x_{i,1},\dots,x_{i,r}]/I^{\sigma_i}$, where I^{σ_i} is given by $I^{\sigma_i} = (f_1^{\sigma_i}(X^{\sigma_i}),\dots,f_s^{\sigma_i}(X^{\sigma_i}))$ and X^{σ_i} is a shorthand for the multivariable $x_{i,1},\dots,x_{i,r}$.

Note that $\overline{A}$ is then given by $M[X^{\sigma_1},\dots,X^{\sigma_n}]/(I^{\sigma_1}+\dots+I^{\sigma_n})$.

The Galois group $G = \mathrm{Gal}(M/K)$ acts on $\overline{A}$ as follows: for every $\tau \in \mathrm{Gal}(M/K)$ and every $i = 1,\dots,n$, the morphism $\tau \circ \sigma_i$ is an embedding of E in $\overline{K}$, hence it is one among the σ_j's. Write j_i for the unique index j such that $\sigma_j = \tau \circ \sigma_i$.

Let τ permute the variables by the rule $\tau \cdot (X^{\sigma_i}) = X^{\sigma_{j_i}}$; together with the natural action of τ on M, this gives $\overline{A}$ the structure of a G-module. We can then consider the K-algebra $\overline{A}^G$, and it is known (see for example [Wei82], pages 4-9) that $\overline{A}^G$ represents $\mathrm{Res}_{E/K}(\Theta)$.

We shall also need a simple lemma:

Lemma 1.1.5.

$$\overline{A}^G \otimes_K M \cong \overline{A}$$

Proof. Fix a base $\alpha_1, \dots, \alpha_n$ of E over K. Then, for each $l = 1, \dots, r$ and each $j = 1, \dots, n$, the element

$$w_{j,l} = \sum_{i=1}^{n} \alpha_j^{\sigma_i} x_{i,l}$$

is stable under the given Galois action: indeed, for any $\tau \in G$, if j_i is again the unique index such that $\tau \circ \sigma_i = \sigma_{j_i}$, we have

$$\tau \left(\sum_{i=1}^{n} \alpha_j^{\sigma_i} x_{i,l} \right) = \sum_{i=1}^{n} \alpha_j^{(\tau \circ \sigma_i)} x_{i,l}^{\tau} = \sum_{i=1}^{n} \alpha_j^{\sigma_{j_i}} x_{j_i,l},$$

and since $i \mapsto j_i$ is simply a permutation, this element is the same as the one we started with.

Now the matrix $S = \left(\alpha_j^{\sigma_i}\right)_{i,j=1,\dots,n}$ is invertible (in M), so every variable $x_{i,l}$ can be written as an M-linear combination of the elements $w_{j,l} \in \overline{A}^G$, i.e., $\overline{A}^G \otimes M \to \overline{A}$ is surjective. □

We can finally prove Theorem 1.1.3:

Proof. By [Hum81], Theorem on page 65, the Lie algebra $\mathfrak{h}$ of Spec $\left(\overline{A}^G\right)$ can be identified to the set of K-derivations on the algebra of regular functions on Spec $\left(\overline{A}^G\right)$, and since this scheme is affine the regular functions are simply $\overline{A}^G$.

By the same result, $\mathfrak{g}$ is the set of E-linear derivations on A; we can therefore define a homomorphism of Lie algebras over E

$$\begin{array}{rccc} \varphi : & Der(A) & \to & Der\left(\overline{A}\right) \\ & \delta & \mapsto & \sum_{i=1}^{n} id \otimes \ldots \otimes id \otimes \underbrace{\delta_i}_{i^{th} \text{ place}} \otimes id \otimes \ldots \otimes id, \end{array}$$

where δ_i is $\delta \otimes_{E,\sigma_i} id \in Der\left(A^{\sigma_i}\right)$.

Observe that if δ is any K-linear derivation and $m \in M$, then m is algebraic over K, hence it admits a minimal separable polynomial $p(x) \in K[x]$. By writing

$$0 = \delta(0) = \delta\left(p(m)\right) = \delta(m) \cdot p'(m),$$

and using $p'(m) \neq 0$ by separability, we get $\delta(m) = 0$, hence δ is M-linear.

To make things a little more explicit, notice that δ_j acts on an element a_j of A^{σ_j}, represented as a polynomial

$$a_j = \sum_{I \text{ multi-index}} m_I \left(X^{\sigma_j}\right)^I,$$

as

$$\delta_j\left(a_j\right)=\sum_I m_I\sigma_j\left(\delta\left(X^I\right)\right)$$

(notice that $\delta\left(X^I\right)\in E$, and σ_j maps E in M).

Let now τ be an element of G, and suppose the indices k_j are chosen in such a way that $\tau\circ\sigma_{k_j}=\sigma_j$. Then $\delta_j\circ\tau\left(a_{k_j}\right)=\tau\circ\delta_{k_j}\left(a_{k_j}\right)$, since for any element

$$a_{k_j}=\sum_{I\text{ multi-index}} m_I\left(X^{\sigma_{k_j}}\right)^I\in A^{\sigma_{k_j}}$$

we have

$$\delta_j\left(\tau\left(\sum_I m_I\left(X^{\sigma_{k_j}}\right)^I\right)\right)=\delta_j\left(\sum_I \tau(m_I)\left(X^{\sigma_j}\right)^I\right)=\sum_I\tau(m_I)\sigma_j\left(\delta\left(X^I\right)\right)$$

and

$$\tau\left(\delta_{k_j}\left(\sum_I m_I\left(X^{\sigma_{k_j}}\right)^I\right)\right)=\tau\left(\sum_I m_I\sigma_{k_j}\left(\delta\left(X^I\right)\right)\right)$$
$$=\sum_I\tau(m_I)(\tau\circ\sigma_{k_j})\left(\delta\left(X^I\right)\right),$$

so the two expressions agree, since $\tau\circ\sigma_{k_j}=\sigma_j$. It immediately follows that $\varphi(\delta)$ is G-equivariant, so $\varphi(\delta)$ induces a derivation on $\overline{A}^G$ and we can consider φ as a map $Der(A)\to Der\left(\overline{A}^G\right)$.

It is also easy to see that $\varphi: Der(A)\to Der\left(\overline{A}\right)$ is injective: let δ be a non-zero derivation on A and a be an element of A such that $\delta(a)\neq 0$. Then $a\otimes_{\sigma_i}1$ is a non-zero element of A^{σ_i} such that $\delta_i(a\otimes 1)\neq 0$, and therefore

$$\varphi(\delta)\left(1\otimes\ldots\otimes 1\otimes\underbrace{(a\otimes_{\sigma_i}1)}_{i^{th}\text{ place}}\otimes 1\otimes\ldots\otimes 1\right)=$$
$$=1\otimes\ldots\otimes 1\ldots\delta_i\left(a\otimes_{\sigma_i}1\right)\otimes 1\otimes\ldots\otimes 1\neq 0,$$

the first equality holding because every derivation δ_j for $j\neq i$ acts on the constant 1, thus yielding zero. Now suppose $\varphi(\delta)$ is trivial on $\overline{A}^G$; by Lemma 1.1.5, we have $\overline{A}^G\otimes M=\overline{A}$, hence by M-linearity $\varphi(\delta)$ is trivial on $\overline{A}$. It follows that $\varphi: Der(A)\to Der\left(\overline{A}^G\right)$ is injective.

Finally, we have

$$\mathrm{Lie}\left(\mathrm{Res}_{E/K}(\Theta)\right)\otimes_K\overline{K}\cong\mathrm{Lie}\left(\mathrm{Res}_{E/K}(\Theta)\otimes_K\overline{K}\right)$$
$$\cong\mathrm{Lie}\left(\Theta^n_{\overline{K}}\right)\cong\left(\mathrm{Lie}\left(\Theta_{\overline{K}}\right)\right)^{\oplus n}$$

$$\cong \left(\mathrm{Lie}\left(\Theta_E\right) \otimes_E \overline{K}\right)^{\oplus n} \cong \mathfrak{g}^{\oplus n} \otimes_E \overline{K},$$

hence

$$\dim_K \mathfrak{h} = \dim_K \left(\mathrm{Lie}\left(\mathrm{Res}_{E/K}(\Theta)\right)\right) = \dim_{\overline{K}} \left(\mathrm{Lie}\left(\mathrm{Res}_{E/K}(\Theta)\right) \otimes_K \overline{K}\right)$$

$$= \dim_{\overline{K}} \left(\mathfrak{g}^{\oplus n} \otimes_E \overline{K}\right) = \dim_E \mathfrak{g}^{\oplus n} = n \dim_E \mathfrak{g} = \dim_K \mathfrak{g},$$

so - comparing dimensions - the injective morphism $\varphi : \mathrm{Res}_{E/K}\, \mathfrak{g} \to \mathfrak{h}$ must be an isomorphism. □

1.1.4 Remarks on the extension-of-scalars functor

Let E be a number field and let V be an E-vector space. We want to study what kind of extra structure $V \otimes_{\mathbb{Q}} \mathbb{C}$ acquires from V being a vector space over E (and not simply over $\mathbb{Q}$).

Let $\Sigma = \Sigma(E)$ denote the set of embeddings $E \hookrightarrow \mathbb{C}$. It is a basic fact from algebraic number theory the existence of an isomorphism

$$\begin{array}{ccc} E \otimes_{\mathbb{Q}} \mathbb{C} & \to & \mathbb{C}^{\Sigma} \\ e \otimes z & \mapsto & (\sigma(e)z)_{\sigma \in \Sigma} \end{array}$$

of vector spaces over E, where the action of E is the obvious one on the left and on the right is given by

$$e \cdot (z_1, \ldots, z_n) = (\sigma_1(e)z_1, \ldots, \sigma_n(e)z_n) \ \forall e \in E, \forall\, (z_1, \ldots, z_n) \in \mathbb{C}^{\Sigma}.$$

Fixing an E-base of V yields an E-linear isomorphism $V \cong E^{\oplus k}$ for a certain k, so we have a chain of isomorphisms of vector spaces

$$V \otimes_{\mathbb{Q}} \mathbb{C} \cong \left(E^{\oplus k}\right) \otimes_{\mathbb{Q}} \mathbb{C} \cong (E \otimes_{\mathbb{Q}} \mathbb{C})^{\oplus k} \cong \left(\mathbb{C}^k\right)^{\Sigma},$$

where, again, E acts on the right through its different embeddings.

An equivalent, coordinate-free construction goes as follows: we have an isomorphism $V \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_\sigma V_\sigma$, where V_σ is the complex vector subspace of $V_{\mathbb{C}}$ given by

$$V_\sigma := \{v \in V_{\mathbb{C}} | e \cdot v = \sigma(e)v \ \ \forall e \in E\}.$$

Note that it follows from the above that all the spaces V_σ are of the same complex dimension. In fact, another equivalent description of the spaces V_σ can be given as follows: V_σ is canonically isomorphic to $V \otimes_{E,\sigma} \mathbb{C}$, the isomorphism being given by

$$\begin{array}{ccc} V \otimes_{E,\sigma} \mathbb{C} & \to & V_\sigma \\ v \otimes z & \mapsto & \pi_\sigma\left(v \otimes 1\right) \cdot z, \end{array}$$

where $\pi_\sigma : V \to V_\sigma$ is the projection.

Suppose now we are given an E-bilinear form $\varphi : V \times V \to E$. We can consider φ as element of $\mathrm{Hom}_E(V \otimes_E V, E) \cong V^* \otimes_E V^*$. By again taking tensor products with $\mathbb{C}$ over E (with respect to a fixed embedding $\sigma : E \hookrightarrow \mathbb{C}$) we get an element $\varphi_\sigma := \varphi \otimes 1 \in V^* \otimes_E V^* \otimes_{E,\sigma} \mathbb{C} \cong (V^* \otimes_{E,\sigma} \mathbb{C}) \otimes_{\mathbb{C}} (V^* \otimes_{E,\sigma} \mathbb{C})$, which again we interpret as a $\mathbb{C}$-bilinear form $V_\sigma \times V_\sigma \to \mathbb{C}$. Moreover, extending scalars between fields carries non-degenerate forms to non-degenerate forms (this can be shown for example by computing determinants).

On the other hand, taking tensor products over $\mathbb{Q}$ yields a bilinear form $\varphi_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to E \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}^\Sigma$, and identifying $V_{\mathbb{C}}$ with $\bigoplus_\sigma V_\sigma$ we see that $\varphi_{\mathbb{C}}$ is simply the collection of the $|\Sigma|$ different forms φ_σ, one on each factor V_σ.

Finally, for our applications it will be useful to observe here that if a $\mathbb{Q}$-algebraic group G acts on V preserving ψ and commuting with E, then clearly $G_{\mathbb{C}}$ acts on $V_{\mathbb{C}}$ preserving $\psi_{\mathbb{C}}$; then

- the spaces V_σ, for various σ's, are $G_{\mathbb{C}}$-stable, since the actions of $G_{\mathbb{C}}$ and E commute (take any $v \in V_\sigma$: then $g \cdot v$ still belongs to V_σ if and only if for every $e \in E$ we have $e \cdot g \cdot v = \sigma(e)(g \cdot v)$, and this is clear, since $g \cdot e \cdot v = g \cdot (\sigma(e)v) = \sigma(e)g \cdot v$, the last two equalities holding since the actions of $G_{\mathbb{C}}$, E and $\mathbb{C}$ commute)

- preserving $\varphi_{\mathbb{C}}$ amounts to preserving each form φ_σ, so each representation V_σ of G comes equipped with a G-invariant form, that is symmetric (resp. skew-symmetric) exactly when φ is.

1.1.5 The Deligne torus

The **Deligne torus** $\mathbb{S}$ is the $\mathbb{R}$-algebraic group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. By definition of the Weil restriction of scalars we have

$$\mathbb{S}(\mathbb{R}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$$

and

$$\mathbb{S}(\mathbb{C}) = \mathbb{G}_{m,\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{C}^* \times \mathbb{C}^*.$$

Definition 1.1.6. The **weight cocharacter** is the cocharacter

$$w : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$$

that induces the natural inclusion $\mathbb{R}^* \hookrightarrow \mathbb{C}^*$ at the level of $\mathbb{R}$ points. More explicitly: $\mathbb{S}$ is represented by the affine algebra $\mathbb{R}\left[a, b, \frac{1}{a^2+b^2}\right]$, $\mathbb{G}_{m.\mathbb{R}}$ is represented by $\mathbb{R}\left[x, \frac{1}{x}\right]$, and w is given at the level of algebras by the unique map sending b to 0 and a to x.

For the Deligne torus $\mathbb{S}$, the $\mathbb{C}$-points are given by $\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^* \cong \mathbb{C}^* \times \mathbb{C}^*$, so the character group of $\mathbb{S}$ is free of rank 2. It is generated by two elements $z, \bar{z}$ that induce respectively the identity and complex conjugation on $\mathbb{R}$-points:

$$\mathbb{C}^* = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C}) \xrightarrow{z,\bar{z}} \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) = \mathbb{C}^*.$$

Remark 1.1.7. To avoid possible confusions, it is better to write down explicitly the characters $z, \bar{z}$ and clarify the identifications among all the objects we will have to deal with. In particular, the identifications we use can be summarized by the following diagram:

$$\begin{array}{ccccccccc} \mathbb{C}^* & \cong & \mathbb{S}(\mathbb{R}) & \hookrightarrow & \mathbb{S}(\mathbb{C}) & \cong & \mathbb{C}^* \times \mathbb{C}^* & \xrightarrow{z} & \mathbb{G}_{m,\mathbb{C}}(\mathbb{C}) = \mathbb{C}^* \\ a+ib & \mapsto & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & \mapsto & \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & \mapsto & (a+ib, a-ib) & \mapsto & a+ib. \end{array}$$

Note furthermore that for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ we can identify

$$\operatorname{Hom}\left(\mathbb{R}\left[a, b, \frac{1}{a^2+b^2}\right], \mathbb{F}\right) \cong GL_2(\mathbb{F})$$

by associating to a morphism φ the matrix $\begin{pmatrix} \varphi(a) & \varphi(b) \\ -\varphi(b) & \varphi(a) \end{pmatrix}$. Finally, observe that

$$\begin{array}{ccc} \operatorname{Hom}\left(\mathbb{R}\left[a, b, \frac{1}{a^2+b^2}\right], \mathbb{C}\right) & \to & \mathbb{C}^* \times \mathbb{C}^* \\ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} & \mapsto & (a+ib, a-ib) \end{array}$$

is well defined, since both $a+ib$ and $a-ib$ are nonzero (as $a^2+b^2 = (a+ib)(a-ib)$ is nonzero).

The **norm character** $\mathrm{Nm} : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ is given by $z\bar{z}$; equivalently, it is the group morphism induced by the algebra map

$$\begin{array}{ccc} \mathbb{R}\left[x, \frac{1}{x}\right] & \to & \mathbb{R}\left[a, b, \frac{1}{a^2+b^2}\right] \\ x & \mapsto & a^2+b^2. \end{array}$$

Finally, define μ to be the only cocharacter

$$\mu : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}$$

such that $\bar{z} \circ \mu \equiv 1$ and $z \circ \mu$ is the identity of $\mathbb{G}_{m,\mathbb{C}}$. μ is given on $\mathbb{C}$-points by $\mu(a) = (a, 1) \in \mathbb{C}^* \times \mathbb{C}^* = \mathbb{S}(\mathbb{C})$.

Remark 1.1.8. Giving a Hodge structure of weight m on a $\mathbb{Q}$-vector space V is equivalent to giving a homomorphism of algebraic groups over $\mathbb{R}$

$$h : \mathbb{S} \to GL(V_{\mathbb{R}})$$

such that $h \circ w : \mathbb{G}_{m,\mathbb{R}} \to GL_{\mathbb{R}}(V)$ is given on $\mathbb{R}$-points by

$$x \in \mathbb{G}_{m,\mathbb{R}}(\mathbb{R}) \mapsto x^{-m} \operatorname{Id}_V \in GL_{\mathbb{R}}(V).$$

If we start with a representation of $\mathbb{S}$ in $GL(V_{\mathbb{R}})$ and extend scalars to $\mathbb{C}$, we obtain

$$V \otimes_{\mathbb{Q}} \mathbb{C} \cong V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{\chi} V_{\chi},$$

where the last sum runs over all characters of $\mathbb{S}_{\mathbb{C}}$ and the (generalized) eigenspace V_{χ} is given by

$$\{v \in V_{\mathbb{C}} | g \cdot v = \chi(g)v \quad \forall g \in \mathbb{S}(\mathbb{C})\}.$$

This last isomorphism arises since every representation of a torus over an algebraically closed field splits as a direct sum of eigenspaces, each one corresponding to a character. Now the characters of $\mathbb{S}_{\mathbb{C}}$ are of the form $z^p \bar{z}^q$, therefore the above decomposition can be rewritten as

$$V \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{(p,q)\in\mathbb{Z}^2} V^{p,q},$$

where (by convention) $V^{p,q}$ is the space of vectors on which $\mathbb{S}_{\mathbb{C}}$ acts through multiplication by $z^{-p}\bar{z}^{-q}$. As the real points x act through multiplication by x^{-m}, we see that a necessary condition for a pair (p, q) to correspond to a non-trivial $V^{p,q}$ is

$$x^{-p}\bar{x}^{-q} = x^{-m} \quad \forall x \in \mathbb{R},$$

i.e. $p + q = m$. As h is defined over $\mathbb{R}$, we have the equality $\tau \cdot h = h$, where τ is the nontrivial automorphism in $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ and it acts on h via $\tau \cdot h := \tau \circ h \circ \tau^{-1} = \tau \circ h \circ \tau$. It follows that if $v \in V^{p,q}$, then

$$h(z)\bar{v} = \tau h(z)\tau^2(v) = \tau h(z)v = \tau \left(z^{-p}\bar{z}^{-q}v\right) = z^{-q}\bar{z}^{-p}\bar{v},$$

so $\overline{V^{p,q}} \subset V^{q,p}$ and $\overline{V^{q,p}} \subset V^{p,q}$. Since complex conjugation induces an automorphism of $V_{\mathbb{C}}$, this readily implies $\overline{V^{p,q}} = V^{q,p}$ for every pair p, q, thus giving the required Hodge structure on V.

On the other hand, if V admits a Hodge structure of weight m, then we get a representation of $\mathbb{S}_{\mathbb{C}}$ on $V_{\mathbb{C}}$ by declaring that it acts through the character $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$. We now need to check that this representation comes from a real representation, but this follows immediately from $\overline{V^{p,q}} = V^{q,p}$ by the same calculation as above. More precisely, we can observe that τ exchanges the two characters of $\mathbb{S}_{\mathbb{C}}$ (since it clearly exchanges them at the level of $\mathbb{R}$-points), and write any $v \in V_{\mathbb{C}}$ as a linear combination of vectors $v^{p,q} \in V^{p,q}$. For any vector w belonging to an eigenspace $V^{p,q}$ we then have

$$h(z_1, z_2)\overline{w} = z(z_1, z_2)^{-q}\bar{z}(z_1, z_2)^{-p}\overline{w} = \overline{z(z_1, z_2)^{-p}\bar{z}(z_1, z_2)^{-q}w} = \tau \circ h(z_1, z_2)w,$$

so that h commutes with τ and $h_{\mathbb{C}}$ comes from a real representation h. Moreover, if x is real (and nonzero), it acts on any one of the spaces $V^{p,q}$ as multiplication by $z(x)^{-p}\bar{z}(x)^{-q} = x^{-p}x^{-q} = x^{-m}$, so it acts on the whole of $V_{\mathbb{C}}$ as multiplication by x^{-m}, as we wanted to show.

Definition 1.1.9. Because of the above Remark, giving a Hodge structure is essentially the same as giving a representation of $\mathbb{S}$. We can therefore carry all the usual notions from representation theory to the setting of Hodge structures: in particular, we can identify Hodge structures with the category of $\mathbb{S}$-representations (over real vector spaces obtained by extension of scalars from rational V's), so that we get natural notions of **dual** structure $V^\vee$, **tensor product**, **morphism** and **sub-Hodge structure**.

Definition 1.1.10. Let V be a Hodge structure of weight n. It is customary to introduce the **Weil operator** C of V,

$$C := h(i) \in GL(V_{\mathbb{R}}).$$

Remark 1.1.11. It follows from the definitions that C acts on $V^{p,q} \subset V_{\mathbb{C}}$ as multiplication by i^{q-p}, hence

$$\begin{aligned} \left(C^2\right)_{\mathbb{C}} &= \bigoplus_{p+q=n} (-1)^{q-p}\,\mathrm{Id}_{V^{p,q}} = \bigoplus_{p+q=n} (-1)^{q+p}\,\mathrm{Id}_{V^{p,q}} = \\ &= \bigoplus_{p+q=n} (-1)^{n}\,\mathrm{Id}_{V^{p,q}} = (-1)^n \left(\mathrm{Id}\right)_{V_{\mathbb{C}}}, \end{aligned}$$

so $C^2 = (-1)^n\,\mathrm{Id}_V$.

Moreover, if V, W are two Hodge structures with Weil operators C_V, C_W, and if $\varphi : V \to W$ is a morphism of Hodge structures, then

$$C_W\varphi(v) = \varphi(C_V v) \quad \forall v \in V_{\mathbb{R}},$$

since φ is h-equivariant (hence $C = h(i)$-equivariant).

1.1.6 Quaternion algebras as algebraic groups

Let E be a number field and D be a quaternion algebra over E. Suppose that D is a skew-field. We then want to interpret D^* as an algebraic group over

E, namely, show that the functor

$$\begin{array}{ccc} \mathrm{Alg}_E & \to & \mathrm{Grp} \\ R & \mapsto & (R \otimes_E D)^* \end{array}$$

is representable by a finitely-generated algebra over E.

By [Mil12] (Remark 3.5 on page 22) it is enough to show that the underlying set functor is representable. By definition of a quaternion algebra, there exist $\alpha, \beta \in D$ such that

1. D is generated by α, β as an E-algebra;
2. $\alpha^2 = a \in E, \beta^2 = b \in E$;
3. $\alpha\beta = -\beta\alpha$
4. $1, \alpha, \beta, \alpha\beta$ is a basis of D over E.

An element γ of D can then be represented uniquely as $\gamma = x \cdot 1 + y \cdot \alpha + z \cdot \beta + w \cdot \alpha\beta$, which suggests that D could be a subscheme of $\mathbb{A}^4_E$. In fact, all we need to do is give a polynomial characterization of 'being a unit', which can be done in terms of the **reduced norm**. Define $N(\gamma) := x^2 - ay^2 - bz^2 + abw^2$ (and $f(x, y, z, w) := x^2 - ay^2 - bz^2 + abw^2$); then γ is a unit if and only if its reduced norm is not zero.

We are thus led to considering the E-algebra

$$B := E[x, y, z, w]_{f(x,y,z,w)} = E[x, y, z, w, u]/(f(x, y, z, w)u - 1);$$

we are going to show that B does represent the desired functor, that is, the equality $\mathrm{Hom}_E(B, R) \cong (R \otimes_E D)^*$ (for every E-algebra R). An E-algebra map from B to R is determined by the images of x, y, z, w (which will again be denoted x, y, z, w), and correspond to the element $x \otimes 1 + y \otimes \alpha + z \otimes \beta + w \otimes \alpha\beta \in R \otimes_E D$. Such an element is invertible, since

$$(x \otimes 1 + y \otimes \alpha + z \otimes \beta + w \otimes \alpha\beta)\,(x \otimes 1 - y \otimes \alpha - z \otimes \beta - w \otimes \alpha\beta) =$$

$$f(x, y, z, w) \otimes 1$$

is invertible; conversely, any invertible element $\gamma' = x' \otimes 1 + y' \otimes \alpha + z' \otimes \beta + w' \otimes \alpha\beta$ of $R \otimes D$ gives rise to a map from B to R by sending x, y, z, w to x', y', z', w'. Note that $f(x', y', z', w')$ is automatically invertible when γ' is: identifying $R \otimes D$ with R^4 via the basis $1, \alpha, \beta, \alpha\beta$, left multiplication by γ' is represented by the matrix

$$M_{\gamma'} = \begin{pmatrix} x' & ay' & -bz' & -abw' \\ y' & x' & bw' & -bz' \\ z' & -aw' & x' & ay' \\ w' & -z' & y' & x' \end{pmatrix},$$

which is invertible exactly when its determinant is. The claim then follows from $\det M_{\gamma'} = f(x', y', z', w')^2$.

1.1.7 Reductive groups

In what will follow we shall make frequent use of the concept of *reductive* (algebraic) group, along with a few basic properties these groups enjoy. We start with some basic definitions:

Definition 1.1.12. An affine group G is **unipotent** if every nontrivial representation V of G admits a nonzero fixed vector, i.e. a v such that

$$g \cdot v = v \quad \forall g \in G.$$

It is possible to show that, given any smooth algebraic group G over a field k, there exists a unique maximal smooth connected normal unipotent subgroup, called the **unipotent radical** R_uG of G. The formation of such R_u commutes with base change, namely for every field extension K/k we have

$$(R_uG)_K = R_u(G_K).$$

Definition 1.1.13. A group G over a field k is **reductive** if it is smooth, connected, and $R_uG_{\bar{k}}$ is trivial.

The Lie algebra $\mathfrak{l}$ of a reductive algebraic group is itself reductive, i.e. it admits a decomposition $\mathfrak{l} \cong \mathfrak{a} \oplus \mathfrak{s}$ with $\mathfrak{a}$ Abelian and $\mathfrak{s}$ semisimple. For a reductive Lie algebra $\mathfrak{l}$ we will write $\mathfrak{l}^{ss}$ for its semisimple part and $\mathfrak{l}^{ab}$ for its Abelian part.

In order to state the basic structure theorem for reductive groups we shall need a few more definitions:

Definition 1.1.14. Let G be an affine group over a field k. Then there exists a unique maximal smooth connected normal solvable subgroup of G, called the **radical** RG of G.

Definition 1.1.15. Let $G_1, G_2, \cdots, G_n$ be subgroups of G. We say that G is the **almost-direct** product of the G_i's, written $G = G_1 \cdot G_2 \cdot \cdots \cdot G_n$, if the natural map

$$\begin{array}{ccc} G_1 \times G_2 \times \cdots \times G_n & \to & G \\ (g_1, g_2, \cdots, g_n) & \mapsto & g_1 g_2 \ldots g_n \end{array}$$

is a surjective homomorphism with finite kernel. In particular, this implies that the G_i's commute with each other and that each of them is normal in G.

Theorem 1.1.16 ([Mil12], Chapter XVII, Theorem 5.1)**.** *Let G be a reductive group, $Z(G)$ be its center, G' be its derived subgroup. Then*

- *G' is semisimple;*
- *the radical of G equals the center of G;*
- *$Z(G)$ is a torus;*
- *G is the almost-direct product $Z(G) \cdot G'$.*

As a corollary we get the following important characterization:

Theorem 1.1.17. *For an algebraic group G over a field k of characteristic zero the following are equivalent:*

1. *G is reductive;*
2. *every representation of G is semisimple;*
3. *G admits at least one faithful semisimple representation.*

Proof. (i) implies (ii) via the above Theorem: indeed, let $G \to GL(V)$ be any representation of $G = Z(G) \cdot G'$. $Z(G)$ is a torus, so - when regarded as a representation of $Z(G)$ - V decomposes as a sum of simple representations,

$$V \cong \bigoplus V_i.$$

$Z(G)$ and G' commute, so each V_i is stable under the action of G', that is to say, it is a G'-module. Moreover, G' is semisimple, hence all its representations are completely reducible, so we can write

$$V_i = \bigoplus_j V_{ij}$$

as a direct sum of simple G'-modules. Then

$$V = \bigoplus_{i,j} V_{ij}$$

is a decomposition of V into a direct sum of simple G-modules.

Clearly (ii) implies (iii), since every algebraic group admits a faithful representation.

Finally, to show that (iii) implies (i), we need one more lemma:

Lemma 1.1.18. *Let G be an algebraic group, V be a semisimple representation of G, U a normal unipotent subgroup. Then U acts trivially on V.*

Proof. If V is the trivial representation there is nothing to show. We can therefore suppose $V \neq (0)$. Let $V^U = \{v \in V | u \cdot v = v \quad \forall u \in U\}$. V^U is nontrivial by definition of unipotent group; it also is a sub-G-module, since - by normality - for every $g \in G$ and for every $u \in U$ we have $ug = gu'$ for a certain $u' \in U$. It follows that

$$u(gv) = g(g^{-1}ug)v = gu'v = gv \quad \forall g \in G, \forall u \in U, \forall v \in V^U,$$

so V^U is a sub-G-module as claimed. As the representation V is semisimple, V^U admits a complement W in V; we want to show that $W = 0$. If, by contradiction, we had $W \neq 0$, then by definition of unipotent group we would also have $W^U \neq 0$, and taking any $w \in W^U \setminus \{0\}$ would give the contradiction $w \in V^U \cap W$. □

Applying the above lemma to a faithful semisimple representation of G we find that any normal unipotent subgroup acts trivially on a faithful representation, hence it is trivial. □

1.2 CM fields

Definition 1.2.1. A **CM-field** is a totally imaginary quadratic extension of a totally real number field E_0, which is called its **maximal totally real subfield**.

Proposition 1.2.2. *Let E be a number field, L its normal closure, $R :=$ $\mathrm{Hom}\,(E, L) = \{\rho_1, \cdots, \rho_n\}$ the set of its embeddings in its normal closure. Fix distinguished embeddings $E \hookrightarrow L \hookrightarrow \mathbb{C}$ and identify $\mathrm{Hom}\,(E, L)$ with $\mathrm{Hom}\,(E, \mathbb{C})$. Denote complex conjugation by τ. Then E is a CM field if and only if the following hold:*

1. *for every $\rho \in R$, $\tau\rho = \rho\tau$*
2. *τ induces a non-trivial automorphism of E*

Proof. Suppose the two conditions hold. Then E_0, the fixed field of E under the action of τ, satisfies $[K : E_0] = 2$ by Artin's lemma. Moreover, E_0 is totally real: every embedding σ of E_0 in $\mathbb{C}$ extends to an embedding ρ of E in $\mathbb{C}$, so for any $\kappa_0 \in E_0$ we have $\tau(\sigma(\kappa_0)) = \sigma(\tau(\kappa_0)) = \sigma(\kappa_0)$ and $\sigma(\kappa_0)$ is real. This shows that E is an imaginary quadratic extension of the totally real field E_0. Moreover, property (1) implies that for every embedding ρ of E in $\mathbb{C}$ τ induces a non-trivial automorphism of E, so no embedding of E can be real and E is totally imaginary, as we wanted to show.

Conversely, suppose E is CM and let E_0 be its maximal totally real subfield. Then E can be written as $E_0[d]$ with $d^2 \in E_0$ totally negative, and τ

is clearly nontrivial on E. Representing an element of E as $\alpha + \beta d$, for any embedding $\rho : E \hookrightarrow \mathbb{C}$ we can write $\rho(\alpha + \beta d) = \sigma(\alpha) + \sigma(\beta)\rho(d)$, where $\sigma = \rho|_{E_0}$. As $\rho(d)^2 \in E_0$ is real but $\rho(d)$ is not, $\tau(\rho(d)) = -\rho(d)$, so

$$\tau\left(\rho(\alpha + \beta d)\right) = \tau(\sigma(\alpha)) + \tau(\sigma(\beta))\tau(\rho(d)) = \sigma(\alpha) - \sigma(\beta)\rho(d)$$

clearly agrees with $\rho(\tau(\alpha + \beta d))$. □

Corollary 1.2.3. *The composite of a finite number of CM-fields is CM; in particular, the normal closure of a CM-field is again a CM-field.*

Proof. The conditions in the above Proposition clearly hold for such a composite field. □

Corollary 1.2.4. *If L is a CM-field, Galois over $\mathbb{Q}$, then complex conjugation is in the center of the Galois group of L.*

Proof. This follows immediately from Proposition 1.2.2, since in this case we can identify the Galois group of L with the set of morphisms $L \hookrightarrow \mathbb{C}$, and we know that complex conjugation commutes with such embeddings. □

1.3 A few properties of Abelian varieties

Throughout this section let A be an Abelian variety of dimension g and $\mathrm{End}(A)$ be its endomorphism ring. We define the **endomorphism algebra** of A, $\mathrm{End}^0(A)$, to be $\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$.

1.3.1 Polarizations

Let $A^\vee$ be the dual variety of A. A **polarization** of A is an isogeny $\lambda : A \to A^\vee$. It is known that, over a field of characteristic zero and up to isogeny, we can always assume A to be isomorphic to $A^\vee$ and choose λ to be an isomorphism: in this case, A is said to be **principally polarized**. Thanks to the canonical identification $A^{\vee\vee} \cong A$, a polarization induces an involution on $\mathrm{End}^0(A)$, the **Rosati involution**, given by

$$\varphi \mapsto \varphi^\dagger := \lambda^{-1} \circ \varphi^\vee \circ \lambda.$$

Note that the Rosati involution is well-defined even if λ is not an isomorphism, since λ is invertible in the category of Abelian varieties up to isogeny.

The Rosati involution enjoys the following properties:

- $(\varphi_1 + \varphi_2)^\dagger = \varphi_1^\dagger + \varphi_2^\dagger \quad \forall \varphi_1, \varphi_2 \in \mathrm{End}^0(A)$;
- $(a\varphi)^\dagger = a\varphi^\dagger \quad \forall \varphi \in \mathrm{End}^0(A), \forall a \in \mathbb{Q}$;

- $(\varphi_1\varphi_2)^\dagger = \varphi_2^\dagger\varphi_1^\dagger \quad \forall\varphi_1,\varphi_2 \in \mathrm{End}^0(A)$;

- $\left(\varphi^\dagger\right)^\dagger = \varphi \quad \forall\varphi \in \mathrm{End}^0(A)$;

- the associated quadratic form $f \mapsto tr\left(ff^\dagger\right)$ is positive-definite ([BL04], Theorem 5.1.8), where tr denotes the reduced trace on the $\mathbb{Q}$-algebra $\mathrm{End}^0(A)$.

Remark 1.3.1. Suppose $F = \mathrm{End}^0(A)$ is a CM field. Then there is a choice of a polarization such that the corresponding Rosati involution induces complex conjugation on F.

To see this, let ι_λ be any Rosati involution, and observe that both ι_λ and complex conjugation are positive involutions, hence they are conjugated under the action of an internal automorphism of $\mathrm{End}^0(A)$ by the Skolem-Noether theorem. The claim then follows because any conjugate of a Rosati involution is again a Rosati involution.

The same statement holds even if F is simply contained in $\mathrm{End}^0(A)$, but it takes a bit more work.

1.3.2 The Albert classification

It is not hard to show that if A is simple, then $\mathrm{End}^0(A)$ is a skew field.

Division algebras with positive involutions have been completely classified by Albert (in [Alb34], [Alb35]); in the context of Abelian varieties, the result is as follows ([Mum70], page 202):

Theorem 1.3.2 (The Albert Classification)**.** *Let $D := \mathrm{End}^0(A)$, L be the center of D (a number field) and L_0 be the subfield of L fixed by the Rosati involution. Let furthermore $e = [L : \mathbb{Q}], e_0 = [L_0 : \mathbb{Q}], d^2 = \dim_L(D)$ and $g = \dim(A)$. Then L_0 is a totally real field, $[L : L_0]$ is either one or two, and if $[L : L_0] = 2$, then L is a CM field. Moreover, in this case it is always possible to choose a polarization λ in such a way that the corresponding Rosati involution is complex conjugation.*

Furthermore, there are only a few possibilities for D, listed in the following table. The fourth column displays numerical constraints e_0, e, d and g must satisfy; these restrictions are actually somewhat different in characteristic $p >$ 0, but we shall only be concerned with the case of characteristic 0.

Type	**e**	**d**		**Description**
I(e_0)	e_0	1	$e_0 \mid g$	$D = L$, a totally real field of degree e_0 over $\mathbb{Q}$
II(e_0)	e_0	2	$2e_0 \mid g$	D is a quaternion algebra over the totally real field F, split at all the infinite places ('totally indefinite quaternion algebra')
III(e_0)	e_0	2	$2e_0 \mid g$	D is a quaternion algebra over the totally real field F, inert at all the infinite places ('totally definite quaternion algebra')
IV(e_0, d)	$2e_0$	any	$e_0 d^2 \mid g$	L is a CM field and D is a division ring of degree d over L

Remark 1.3.3. A result of Shimura ([Shi63]) implies that all the above actually appear as endomorphism algebras of Abelian varieties.

If one asks the subtler question of whether there is a *simple* Abelian variety A with a given endomorphism algebra, the answer becomes slightly more complicated. It turns out that there always exists such an A, except for types III and IV, when the quotient $g/(2e)$ in the first case and $\frac{g}{2ed^2}$ in the second case is either 1 or 2 (see [Mum70], page 203); even in this case, a complete classification result is available (and is again due to Shimura).

In the case of Abelian surfaces, for example, the center of the endomorphism algebra can never contain an imaginary quadratic extension of $\mathbb{Q}$, and Type III does not arise.

CHAPTER 2

A little representation theory

We recall a few standard definitions and results in the theory of (semi)simple Lie algebras that will turn out to be of utmost importance in the study of higher-dimensional Mumford-Tate groups. The main references for this section are [Ser01], [Bou08] and [Hum73].

2.1 Definitions

Definition 2.1.1. Let V be a finite-dimensional vector space over $\mathbb{R}$ and α be an element of V. A **symmetry** with vector α is any linear automorphism s of V satisfying:

- $s(\alpha) = -\alpha$;
- the set $\{v \in V | s(v) = v\}$ of fixed points of s is a hyperplane of V.

Clearly, a symmetry with vector α is far from being unique; there is, however, a simple additional condition under which s is determined by α:

Lemma 2.1.2. *Suppose R is a finite subset of V that spans it. Then there is at most one symmetry with vector α that leaves R invariant.*

Proof. Let s_1, s_2 be two symmetries with the above property. Let $s = s_1 s_2^{-1}$. Then on one hand s induces a permutation of R, and since this is a finite set there exists a certain $n \in \mathbb{N}$ such that s^n is the identity on R, and hence on V. On the other hand, s acts as the identity on both span(α) and $V/\mathbb{R}\alpha$, so its only eigenvalue is 1. This implies that the minimal polynomial of s divides both $x^n - 1$ and $(x-1)^k$ for a sufficiently large k, hence it divides $x - 1$, which forces $s = \mathrm{Id}_V$. □

Definition 2.1.3. A finite subset R of a real, finite-dimensional vector space V is said to be a **root system (in V)** if the following conditions are satisfied:

(R1) R does not contain 0 and it spans V;

(R2) for every $\alpha \in R$, there exists a symmetry with vector α leaving R invariant. The above Lemma implies that such a symmetry is unique, so it makes sense to write s_α for this symmetry.

(R3) for any two roots $\alpha, \beta \in R$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

The dimension of V is called the **rank** of V.

Remark 2.1.4. As R is stable under s_α we immediately see that, for every root α, $s_\alpha(\alpha) = -\alpha$ is again a root. On the other hand, suppose $\alpha, t\alpha$ are both in R for a certain $0 < t < 1$. Then $s_\alpha(t\alpha) - t\alpha = -2t\alpha$ is an integer multiple of α by condition (R3), so $2t \in \mathbb{Z}$ and $t = \frac{1}{2}$.

A root system is said to be **reduced** if, for every $\alpha \in R$, $\pm\alpha$ are the only roots proportional to α; reduced systems are exactly the ones arising in the context of semisimple Lie algebras over algebraically closed fields.

Definition 2.1.5. The group W generated by the reflections s_α for α varying in R is called the **Weyl group** W of the root system, and plays a very important role in the theory of semisimple Lie algebras.

Remark 2.1.6. Any element w of the Weyl group can be thought of as a permutation of R, thus giving rise to a group morphism $W \to \mathcal{S}_R$: this map is an injection, since R spans V, and the latter group is finite, since R is. This shows that W is finite.

Proposition 2.1.7. *There exists a positive-defined, symmetric, W-invariant bilinear form on V.*

Proof. Fix any positive-defined symmetric bilinear form B on V. Then

$$(x, y) := \sum_{w \in W} B(wx, wy) \quad \forall x, y \in V$$

is clearly positive-defined, and it is W-invariant since

$$(gx, gy) = \sum_{w \in W} B(wgx, wgy) = \sum_{w \in Wg^{-1}} B(wgx, wgy) =$$

$$\sum_{w \in W} B(wx, wy) = (x, y) \quad \forall g \in W, \forall x, y \in V.$$

□

From now on, $(\cdot,\cdot)$ will denote such a bilinear form and $||\cdot||$ will denote the associated W-invariant norm on V. This data gives V the structure of a Euclidean space.

Remark 2.1.8. We know from the definition (property R3) that $s_\alpha(x)-x=n\alpha$ for a certain integer n. The W-invariance of $(\cdot,\cdot)$ implies

$$(x,\alpha)=(s_\alpha(x),s_\alpha(\alpha))=(x+n\alpha,-\alpha)=-(x,\alpha)-n(\alpha,\alpha),$$

so $n=-2\frac{(x,\alpha)}{(\alpha,\alpha)}$. We then obtain a integer-valued form $\langle y,x\rangle=2\frac{(x,y)}{(y,y)}$. This form is not symmetric and is linear only in the second argument.

Remark 2.1.9. As $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$ is an integer, writing $(\alpha,\beta)=||\alpha||\cdot||\beta||\cos(\varphi)$ we get $4\cos(\varphi)^2\in\mathbb{Z}$, which gives a finite number of possibilities for φ. Listing such possibilities shows that either $\langle\alpha,\beta\rangle=\langle\beta,\alpha\rangle=0$ or

$$\min\{|\langle\alpha,\beta\rangle|,|\langle\alpha,\beta\rangle|\}\leq 1.$$

Lemma 2.1.10. *If α,β are roots and $\langle\alpha,\beta\rangle>0$, then $\alpha-\beta$ is again a root.*

Proof. Combining the previous remark, the hypothesis and the obvious inequality $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle\geq 0$ we see that both $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ are positive. The last statement in the previous remark implies that at least one of these values is 1; since $\langle\alpha,\beta\rangle>0$ is equivalent to $\langle\beta,\alpha\rangle>0$ and $\alpha-\beta$ is a root if and only if $\beta-\alpha$ is, without loss of generality we can assume $\langle\alpha,\beta\rangle=1$. It follows that R, being stable under both s_α and central symmetry, contains $-s_\alpha(\beta)=-(\beta-\langle\alpha,\beta\rangle\alpha)=\alpha-\beta$. □

The previous lemma, while useful in itself, will also be crucial in order to justify the following definition of direct sum of root systems.

Definition 2.1.11. Let R be a root system in V, and suppose V can be written as a direct sum $V_1\oplus V_2$ in such a way that $R\subset V_1\cup V_2$. Let R_i be the intersection $R\cap V_i$: V is then said to be the **direct sum** of (V_1,R_1) and (V_2,R_2).

If V can be written as a direct sum only trivially - i.e. with $V_1=\{0\}$ or $V_2=\{0\}$ - then it is said to be **irreducible**.

In order for the previous Definition to make sense, we need to check that each (V_i,R_i) is a root system. This is true, and moreover V_1 is automatically orthogonal to V_2 with respect to any W-invariant symmetric bilinear form on V.

Proof. Clearly a root α belongs to R_i if and only if $-\alpha$ does. Let now α (resp. β) be an element of R_1 (resp. R_2). Since $\alpha-\beta$ does not belong to $V_1\cup V_2$, it does not belong to R, hence the previous Lemma implies $(\alpha,\beta)\leq 0$. Since the same applies to $\alpha,-\beta$ we find that $(\alpha,\beta)=0$.

Let W_1, W_2 be the spans of R_1, R_2 respectively. W_1 and W_2 are orthogonal by the above, and on the other hand their sum is all of V, since R spans V. We deduce $V = W_1 \oplus W_2$, and since $W_i \subseteq V_i$ we get $W_i = V_i$, so R_i spans V_i and V_1 is orthogonal to V_2.

Finally, for any $\alpha \in R_1$ the symmetry s_α preserves R_2 (since for any $\beta \in R_2$ we have $s_\alpha(\beta) = \beta - \langle \alpha, \beta \rangle \alpha = \beta$), hence it preserves V_2. As V_1 is the orthogonal complement of V_2 in V and s_α is an isometry for the given scalar product, s_α must preserve V_1, hence $s_\alpha(R_1) \subset R \cap V_1 = R_1$. This completes the verification that (V_i, R_i) is a root system. □

Definition 2.1.12. A **base** of a root system (V, R) is a subset $\Delta = \{\alpha_1, ..., \alpha_l\}$ of R that is a basis for V and such that every $\alpha \in R$ can be written (uniquely, since Δ is a base) as a combination of the α_i's with integral coefficients, all non-negative or non-positive.

Even though it is not immediate from the definition, a base always exists, and moreover the set of all the bases is acted upon by the Weyl group in a simply transitive fashion. A choice of Δ induces a partial ordering on R, given by $\lambda \succ \mu$ if and only if $\lambda - \mu$ can be written as a non-negative combination of elements of Δ. Furthermore, it turns out that giving a base Δ is equivalent to giving a set of **positive roots** R^+, i.e. a subset of R with the following two properties:

- $\alpha \in R^+ \Leftrightarrow -\alpha \notin R^+$;
- If $\alpha, \beta \in R^+$ and $\alpha + \beta \in R$, then $\alpha + \beta \in R^+$.

The correspondence between bases and positive roots is given as follows:

- given a base Δ, an admissible set of positive roots is given by those roots that can be written as a non-negative linear combination of elements in Δ;
- given a choice for R^+, a base is given by the set of positive roots that cannot be written as the sum of two positive roots (the so-called **simple** roots).

Bases turn out to be closely related to the following geometric object:

Definition 2.1.13. For every $\alpha \in R$ let P_α be the hyperplane fixed by s_α.

Then $V \setminus \cup_{\alpha \in R} P_\alpha$ has a finite number of connected components, called the **Weyl chambers** of the root system.

It is possible to show that bases are in 1-to-1 correspondence with Weyl chambers, and in particular the action of W on the Weyl chambers is simply transitive. This implies that for any fixed chamber C, there is a unique $w_0 \in W$, called the **opposition involution** (with respect to the given chamber), sending C to $-C$. It can happen that the opposition involution is simply given by $x \mapsto -x$, but this is not always the case.

In what follows, 'opposition involution' will always mean the one associated to the Weyl chamber pertaining to a given base $\Delta = \{\alpha_1, \dots, \alpha_l\}$.

For irreducible root systems the ordering induced by a base has the following useful property:

Proposition 2.1.14. *Let (V, R) be an irreducible root system and fix a base Δ. Then the ordering induced by Δ on R admits a unique maximal element H, which is called the* ***longest root*** *of R.*

Writing $H = \sum_{\alpha\in\Delta} c_\alpha \alpha$ for the expression of H in terms of the base we also have $c_\alpha \geq 1 \, \forall \alpha \in \Delta$.

It is also customary to introduce the notion of **coroot**: for any $\alpha \in R$, the coroot $\alpha^\vee$ is defined to be $\alpha^\vee := \frac{2}{(\alpha,\alpha)}\alpha$.

Remark 2.1.15. Note that $(\beta, \alpha^\vee) = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = \langle \alpha, \beta \rangle$.

Given a root system (V, R), the set $R^\vee = \{\alpha^\vee | \alpha \in R\}$ again defines a root system $(V, R^\vee)$, called the **dual root system**, with Weyl group canonically isomorphic to the Weyl group of (V, R). In order to classify representations of simple Lie algebras (that are naturally associated with root systems) it is useful to introduce the notion of **weight**: a weight for the root system (V, R) is a $\lambda \in V$ such that $(\lambda, \alpha^\vee)$ is an integer for every root α. The weights form a lattice in V, called the **weight lattice** $P(R)$ of (V, R).

The ordering on the roots gives rise to a notion of 'positivity' on the weights as follows: fix a base $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of R. Then a weight $\lambda \in P(R)$ is said to be **dominant** if

$$(\lambda, \alpha^\vee) \geq 0 \quad \forall \alpha \in \Delta,$$

or, equivalently, if the same inequality holds for every root α positive with respect to the order induced by Δ.

As $\{\alpha^\vee | \alpha \in \Delta\}$ is a basis of V, it admits a dual basis ω_j with respect to $(\cdot, \cdot)$; it is clear from the definitions that these ω_j's are weights and that they are in fact dominant. We then define

$$P_{++}(R) := \bigoplus_{i=1}^{l} \mathbb{N}\omega_i,$$

the set of **dominant weights** of (V, R), and we call the ω_j's the **fundamental dominant weights** (with respect to the given choice of positive roots).

Finally, the **Cartan matrix** of the root system (V, R) (relative to a base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$) is $C_{ij} := \langle \alpha_i, \alpha_j \rangle$.

Note that the definition of the Cartan matrix depends both on Δ and on the ordering of the elements of the base; while changing the ordering of the elements of Δ amounts to conjugating the Cartan matrix by a permutation matrix, choosing a different base does *not* change the Cartan matrix (up to permutation), since any two bases are conjugated under the Weyl group and the numbers $\langle \alpha, \beta \rangle$, being ratios of scalar products, are invariant under the action of the Weyl group.

It can be shown that a root system can be reconstructed (up to isomorphism) from its Cartan matrix, which turns out to always be nonsingular.

Remark 2.1.16. From the fact that the Cartan matrix has integer coefficients we can easily prove that each ω_i is a linear combination of $\alpha_1, \ldots, \alpha_l$ with *rational* coefficients: write $\omega_j = \sum_{i=1}^{l} d_{ji} \alpha_i$ for the expression of ω in terms of the base. Then by definition of a dual basis we have

$$\delta_{jk} = \left(\omega_j, \alpha_k^\vee\right) = \sum_{i=1}^{l} d_{ji} \left(\alpha_i, \alpha_k^\vee\right) = \sum_{i=1}^{k} d_{ji} \langle \alpha_k, \alpha_i \rangle = \sum_{i=1}^{k} d_{ji} C_{ki},$$

or, in matrix terms, $\mathrm{Id} = DC^t$, where D is the matrix whose entries are the d_{ji}'s and C is the Cartan matrix. It then follows from Cramer's formula that $D = \left(C^t\right)^{-1}$ has rational coefficients, and in fact the unique denominator appearing is the determinant of C.

2.2 Root systems and representation theory

Let $\mathfrak{g}$ be a (nonzero) semisimple Lie algebra over an algebraically closed field (of characteristic zero). It can be shown that $\mathfrak{g}$ admits non-zero subalgebras consisting entirely of semisimple elements; any such subalgebra is said to be **toral**, and is automatically Abelian. A maximal toral subalgebra of $\mathfrak{g}$ is called a **Cartan subalgebra**. Fix such a subalgebra H. The **adjoint representation** of $\mathfrak{g}$ is given by

$$\begin{aligned} \mathfrak{g} &\to \mathrm{End}\,(\mathfrak{g}) \\ g &\mapsto [g, \cdot]\,, \end{aligned}$$

and it can be shown that $ad(h)$ is diagonalizable for every h in H. Moreover, since H is Abelian, so is $ad(H)$, which implies - by a standard result in linear algebra - that all the endomorphisms in $ad(H)$ can be diagonalized simultaneously. $\mathfrak{g}$ can therefore be written as the direct sum of a certain number of common generalized eigenspaces $\mathfrak{g}_\lambda := \{x \in \mathfrak{g} | [h, x] = \lambda(h) x \; \forall h \in H\}$, where λ is a linear functional on H.

Note that $\mathfrak{g}_0$ is simply the centralizer of H in $\mathfrak{g}$, which can be shown to equal H. Let R be the set of λ's that correspond to non-trivial generalized eigenspaces. We then have the decomposition

$$\mathfrak{g} \cong H \oplus \bigoplus_{\lambda \in R} \mathfrak{g}_\lambda.$$

If we let $E_{\mathbb{Q}}$ be the $\mathbb{Q}$-subspace of $H^\vee$ spanned by R, then $R \otimes 1$ is a root system in $E = E_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$; moreover, two semisimple Lie algebras corresponding to isomorphic root system are isomorphic. In particular, the decomposition of a semisimple Lie algebra as direct product of its simple factors corresponds to the decomposition of the associated root system as direct sum of irreducible systems. In turn, irreducible root systems have been classified completely, and - apart from five exceptional cases - fall into four infinite families.

The correspondence between root systems and simple Lie algebras is described by the following table, where the subscript in the name of the root system is the rank and the inequalities are necessary in order to avoid repetitions (there are some so-called 'exceptional isomorphisms' between Lie algebras of low rank)

Lie algebra	**Root system**	$\mathbf{|R|}$	**Weyl group**
$\mathfrak{sl}_{l+1}$, $l \geq 1$	A_l	$n(n+1)$	$\mathcal{S}_{l+1}$
$\mathfrak{so}_{2l+1}$, $l \geq 2$	B_l	$2n^2$	$(\mathbb{Z}/2\mathbb{Z})^l \rtimes \mathcal{S}_l$
$\mathfrak{sp}_{2l}$, $l \geq 3$	C_l	$2n^2$	$(\mathbb{Z}/2\mathbb{Z})^l \rtimes \mathcal{S}_l$
$\mathfrak{so}_{2l}$, $l \geq 4$	D_l	$2n(n-1)$	$(\mathbb{Z}/2\mathbb{Z})^{l-1} \rtimes \mathcal{S}_l$
$\mathfrak{e}_6$	E_6	72	Of order $2^7\,3^4\,5$
$\mathfrak{e}_7$	E_7	126	Of order $2^{10}\,3^4\,5\,7$
$\mathfrak{e}_8$	E_8	240	Of order $2^{14}\,3^5\,5^2\,7$
$\mathfrak{f}_4$	F_4	48	Of order $2^7\,3^2$
$\mathfrak{g}_2$	G_2	12	D_6 (dihedral group)

The four infinite families of simple Lie algebras can be easily described as matrix algebras with the Lie bracket being given by the usual commutator:

- $\mathfrak{sl}_{l+1}$: traceless matrices of rank $l+1$.
- $\mathfrak{so}_{2l}$, $\mathfrak{so}_{2l+1}$: skew-symmetric matrices of rank $2l$, $2l+1$.
- $\mathfrak{sp}_{2n}$: let J be the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ identity matrix. Then $\mathfrak{sp}_{2n}$ is the set of $2n \times 2n$ matrices A such that $JA + A^T J = 0$.

These four families are collectively called 'the classical algebras', and their associated root systems are said to be 'of classical type'.

We now turn to the classification of representations of semisimple Lie algebras over $\mathbb{C}$ (or, more generally, an algebraically closed field of characteristic zero). Let $\mathfrak{g}$ be a semisimple Lie algebra, H a fixed Cartan subalgebra, R the associated root system, $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ a base of R and W the Weyl group. Let V be a module over $\mathfrak{g}$. Restricting the given action to H we get the notion of **weight spaces** V_λ, where λ is a linear functional on H:

$$V_\lambda = \{v \in V | h \cdot v = \lambda(h)v \;\; \forall h \in H\}\,.$$

Those λ's such that V_λ is non-trivial are traditionally called the **weights** of the representation. The sum $V' := \sum_{\lambda \in H^\vee} V_\lambda$ is always direct, and V' is a submodule of V; furthermore, if V has finite dimension, then $V = V'$. Moreover, Weyl's complete reducibility theorem implies that every finite-dimensional $\mathfrak{g}$-module is completely reducible, hence irreducible representations play a prominent role in the finite-dimensional case.

Definition 2.2.1. Recall that we have a decomposition $\mathfrak{g} \cong H \oplus \bigoplus_{\lambda \in R} \mathfrak{g}_\lambda$. The **Borel subalgebra** of $\mathfrak{g}$ (associated to H and Δ) is

$$B := H \oplus \bigoplus_{\lambda \in R^+} \mathfrak{g}_\lambda;$$

its derived algebra is $N = \bigoplus_{\lambda \in R^+} \mathfrak{g}_\lambda$.

A **highest weight vector** of V is a non-zero vector v belonging to a weight space V_λ killed by all the elements in N, i.e., such that $x \cdot v = 0 \; \forall x \in N$. In this case we also say that λ is a **highest weight** of V, and if V is generated by v as a $\mathfrak{g}$-module, then we say that V is **standard cyclic of weight** λ.

The terminology 'highest weight' is justified by the following theorem:

Theorem 2.2.2. *Let V be a standard cyclic $\mathfrak{g}$-module of highest weight vector $v \in V_\lambda$. Then:*

- *V is the direct sum of its weight spaces;*
- *the weights of V are of the form $\lambda - \sum_{i=1}^{l} k_i \alpha_i$, where each k_i is a non-negative integer;*
- *each weight space V_μ is finite-dimensional and $\dim V_\lambda = 1$;*
- *each submodule of V is the direct sum of its weight spaces;*
- *V is indecomposable (i.e. it cannot be written as the direct sum of two proper submodules); it admits a unique maximal proper submodule and a corresponding unique irreducible quotient;*

- *every nonzero homomorphic image of V is again standard cyclic of weight λ.*

The main results concerning existence and uniqueness of representations are as follows:

Theorem 2.2.3. *1. Every finite-dimensional $\mathfrak{g}$-module admits a highest weight vector.*

2. Let V be standard cyclic of weight λ and suppose furthermore that V is irreducible. Then there is (up to scalar multiples) only one highest weight vector.

Combining this with the previous point, we see that every irreducible representation (of finite dimension) is standard cyclic and admits a unique highest weight.

3. If V_1, V_2 are standard cyclic of weight λ and irreducible, then they are isomorphic.

4. For each $\lambda \in H^\vee$ there exists a (unique) standard cyclic irreducible module of highest weight λ, denoted $V(\lambda)$.

5. The map $\lambda \mapsto V(\lambda)$ establishes a bijection between the set of dominant weights of H and the set of isomorphism classes of finite-dimensional, irreducible $\mathfrak{g}$-modules.

2.3 Self-dual representations

We briefly recall a few results on self-duality properties for Lie algebras representations over the complex numbers (or more generally an algebraically closed field $\mathbb{F}$ of characteristic zero). Recall that if $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation, than the dual representation of ρ is defined by the action

$$(g \cdot \psi)(v) = -\psi(\rho(g)v) \quad \forall \psi \in V^*, \forall v \in V, \forall g \in \mathfrak{g}.$$

A representation is said to be **self-dual** if it is isomorphic to its own dual. A finer notion is given by the following

Definition 2.3.1. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a Lie algebra representation, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ be a bilinear form on V. $\langle \cdot, \cdot \rangle$ is said to be **ρ-invariant**, or **preserved by** ρ, if

$$\langle gv, w \rangle + \langle v, gw \rangle = 0 \quad \forall v, w \in V, \forall g \in \mathfrak{g}.$$

A representation ρ is said to be **orthogonal** (resp. **symplectic**) if there exists a non-degenerate, symmetric (resp. skew-symmetric) bilinear form preserved by ρ.

An immediate consequence of the definition is that orthogonal and symplectic representations are automatically self-dual: if $\langle\cdot,\cdot\rangle$ is a ρ-invariant, non-degenerate bilinear form, then the map

$$\begin{array}{rccc}\varphi: & V & \to & V^\vee \\ & v & \mapsto & \langle v,\cdot\rangle\end{array}$$

is an isomorphism of $\mathfrak{g}$-representations. Indeed, it is clear that φ is an isomorphism of vector spaces, and the equality

$$\varphi(gv) = \langle gv,\cdot\rangle = -\langle v, g\cdot\rangle = g\cdot(\langle v,\cdot\rangle) = g\varphi(v)$$

shows that it is a morphism of representations. Over the algebraically closed field $\mathbb{F}$, Schur's lemma implies that every irreducible, self-dual representation admits a non-trivial, invariant bilinear form:

Lemma 2.3.2. *A finite-dimensional irreducible Lie algebra representation over $\mathbb{F}$ is self-dual if and only if it is either orthogonal or symplectic (but not both).*

Proof. We have already shown the 'if' part. For the other implication, let $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$ be a finite dimensional, self-dual representation and $\varphi: V\to V^\vee$ be a $\mathfrak{g}$-isomorphism. Then the bilinear form

$$\langle v,w\rangle_\varphi := \varphi(v)(w)$$

is $\mathfrak{g}$-invariant, since

$$\langle gv,w\rangle + \langle v,gw\rangle = \varphi(gv)(w) + \varphi(v)(gw) =$$

$$= (g\cdot\varphi(v))(w) + \varphi(v)(gw) = -\varphi(v)(gw) + \varphi(v)(gw) = 0.$$

In fact, it is easy to check that the space $\mathrm{Hom}_{\mathfrak{g}}(V,V^\vee)$ and the space $B_{\mathfrak{g}}(V)$ of $\mathfrak{g}$-invariant bilinear forms on V are isomorphic to each other via the above association $\varphi\mapsto\langle\cdot,\cdot\rangle_\varphi$. If V is irreducible and isomorphic to its dual, then clearly $V^\vee$ is irreducible, too, and Schur's lemma applies to yield

$$\dim_{\mathbb{F}}(B_{\mathfrak{g}}(V)) = \dim_{\mathbb{F}}\left(\mathrm{Hom}_{\mathfrak{g}}\left(V,V^\vee\right)\right) = 1.$$

In particular, every $\mathfrak{g}$-invariant bilinear form that is not trivial is non-degenerate, since it induces an isomorphism $V\to V^\vee$. Let now $b(v,w)$ be any non-zero, $\mathfrak{g}$-invariant bilinear form. Define $b^*(v,w) := b(w,v)$. By the above, there exists $c\in\mathbb{F}$ such that $b^* = cb$. Then

$$b(v,w) = b^*(w,v) = cb(w,v) = cb^*(v,w) = c^2 b(v,w) \quad \forall v,w\in V,$$

hence $c=\pm1$ and b is either symmmetric or skew-symmetric. Moreover, by non-degeneracy, it cannot be both, unless $\mathrm{char}(\mathbb{F}) = 2$, when the two notions coincide.

□

We now turn to studying self-dual representations of semi-simple algebras. We first need a definition:

Definition 2.3.3. Let $\rho_i : \mathfrak{g}_i \to \mathfrak{gl}(V_i)$ be a finite family of representations of Lie algebras $\mathfrak{g}_i$. The **exterior tensor product** of the ρ_i's, denoted

$$\rho_1 \boxtimes \cdots \boxtimes \rho_n \quad \text{(or } V_1 \boxtimes \cdots \boxtimes V_n \text{ when no confusion can arise)},$$

is a representation of the product algebra $\prod_{i=1}^{n} \mathfrak{g}_1$ on the vector space $V_1 \otimes \cdots \otimes V_n$, the action being given (on the generators) by

$$(g_1, \ldots, g_n) \cdot (v_1 \otimes \cdots v_n) = \sum_{i=1}^{n} v_1 \otimes \cdots \otimes v_{i-1} \otimes \rho_i(g_i)(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_n.$$

In order to study generic self-dual representations we first state the following structure theorem:

Theorem 2.3.4. *For a semisimple Lie algebra $\mathfrak{k}$ denote by $\mathfrak{R}(\mathfrak{k})$ the set of isomorphism classes of irreducible representations of $\mathfrak{k}$. Let $\mathfrak{g} \cong \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ be a semisimple Lie algebra, decomposed as a product of simple Lie algebras. Then the map*

$$\begin{array}{cccc} \Psi : & \prod_{i=1}^{n} \mathfrak{R}(\mathfrak{g}_i) & \to & \mathfrak{R}(\mathfrak{g}) \\ & (V_1, \rho_1), \cdots, (V_n, \rho_n) & \mapsto & \rho_1 \boxtimes \cdots \boxtimes \rho_n \end{array}$$

is a bijection.

Remark 2.3.5. In particular, the order of the factors V_i is important: if V_1, V_2 are representations of the same algebra $\mathfrak{g}$, then the two representations $V_1 \boxtimes V_2$ and $V_2 \boxtimes V_1$ of $\mathfrak{g} \times \mathfrak{g}$ are not isomorphic unless $V_1 \cong V_2$.

Proof. This is an almost immediate consequence of the fact that every irreducible representation of a semi-simple Lie algebra is generated by a highest weight vector. Write $V_i(\mu)$ for the irreducible representation of $\mathfrak{g}_i$ with highest weight μ, and similarly let $V(\mu)$ be the irreducible representation of $\mathfrak{g}$ with highest weight μ.

Fix Cartan sub-algebras $\mathfrak{h}_i$ of $\mathfrak{g}_i$, and consider the associated root systems $(\mathfrak{h}_i^\vee, \Phi_i)$. Clearly, $\bigoplus_{i=1}^{n} \mathfrak{h}_i$ is a Cartan sub-algebra of $\mathfrak{g}$, and a choice of bases for the root systems Φ_i induces a base for the root system associated to $\mathfrak{h}$.

Observe now that each weight λ of $\mathfrak{h}$ can be written uniquely as $\lambda = \sum_{i=1}^{n} \lambda_i$, where each λ_i is a weight of the corresponding $\mathfrak{h}_i$. It is then easy to check that $V(\lambda)$ is the exterior tensor product $V(\lambda_1) \boxtimes \cdots \boxtimes V(\lambda_n)$: such a product representation is automatically irreducible, so - in order to prove that it is isomorphic to $V(\lambda)$ - it is enough to exhibit a highest weight vector of weight λ. Such a vector is in fact given by $v_1 \otimes \cdots \otimes v_n$, where each v_i is a highest weight vector for the corresponding $V(\lambda_i)$, and so Ψ is surjective.

To show injectivity, observe that - with the above notation - $v_1 \otimes \cdots \otimes v_n$ is a highest weight vector for $\Psi(V(\lambda_1), \cdots, V(\lambda_n))$.

$\Psi(V(\lambda_1), \cdots, V(\lambda_n))$ is irreducible, so it has a unique line of such vectors: this means that we can recover $\lambda_1, \cdots, \lambda_n$ from $\Psi(V(\lambda_1), \cdots, V(\lambda_n))$ by taking a highest weight vector $v \in \Psi(V(\lambda_1), \cdots, V(\lambda_n))$, computing its weight λ and decomposing it along the various $\mathfrak{h}_i$'s. This shows that ψ is injective and concludes the proof. □

Corollary 2.3.6. *Let $\mathfrak{g} \cong \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ be a semisimple Lie algebra, written as the direct product of its simple factors, and let V be an irreducible representation of $\mathfrak{g}$. Then, by the above theorem, we have a decomposition $V \cong W_1 \boxtimes \cdots \boxtimes W_n$. Suppose that V is self-dual: then each W_i is self-dual.*

Proof. We have $V \cong V^\vee \cong W_1^\vee \boxtimes \cdots \boxtimes W_n^\vee$, so

$$\Psi(W_1, \cdots, W_n) \cong \Psi\left(W_1^\vee, \cdots, W_n^\vee\right),$$

and as Ψ is injective (on n-uples of isomorphism classes) we get $W_i \cong W_i^\vee$ for all i. □

2.4 Minuscule weights

We now want to introduce the special class of **minuscule** weights, that will play a prominent role in what will follow. As a first step we need the notion of R-saturated set:

Definition 2.4.1. A subset X of $P(R)$ is said to be **R-saturated** if for every $\lambda \in X$, for every root α and every integer i between 0 and $(\lambda, \alpha^\vee)$ the weight $\lambda - i\alpha$ belongs to X.

Remark 2.4.2. A moment's thought shows that a R-saturated set is automatically W-invariant, since elementary reflections w_α send λ to $\lambda - (\lambda, \alpha^\vee)\,\alpha$.

Proposition 2.4.3. *Let $\lambda \in P(R)$ be a weight, $X(\lambda)$ the smallest R-saturated subset of $P(R)$ containing λ. Recall that V has the structure of a Euclidean space with respect to a norm $||\cdot||$ that is W-invariant.*

Then the following are equivalent:

1. $X(\lambda) = W \cdot \lambda$

2. *All the elements in $X(\lambda)$ have the same norm*

3. *For every $\alpha \in R$, $||\lambda|| \leq ||\lambda - t\alpha||$ for every integer t between 0 and $(\lambda, \alpha^\vee)$*

4. $(\lambda, \alpha^\vee) \in \{-1, 0, 1\} \quad \forall \alpha \in R$.

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial from the definitions.

As for the implication (3) ⇒ (4), fix $\alpha \in R$ and consider the function $f(t) : t \mapsto ||\lambda - t\alpha||^2$. $f(t)$ is strictly convex, since α is a root (hence non-zero). Notice furthermore that

$$||\lambda|| = ||s_\alpha(\lambda)|| = ||\lambda - \langle \alpha, \lambda \rangle \alpha|| = ||\lambda - \left(\lambda, \alpha^\vee\right) \alpha||,$$

so $f\left((\lambda, \alpha^\vee)\right) = f(0)$. For any s strictly between 0 and $(\lambda, \alpha^\vee)$ we can write $s = t \cdot 0 + (1-t) \cdot (\lambda, \alpha^\vee)$ for a certain $t \in (0, 1)$; convexity then yields

$$f(s) < tf(0) + (1-t) f\left(\left(\lambda, \alpha^\vee\right)\right) = f(0).$$

If $|\left(\lambda, \alpha^\vee\right)|$ is greater than one, then there exists an integer s in the interval $(0, (\lambda, \alpha^\vee))$ (resp. $(\left(\lambda, \alpha^\vee\right), 0)$, if $(\lambda, \alpha^\vee) < 0$), and the above inequality - applied to this s - contradicts the hypothesis (3), so (4) must hold.

Finally, assume that (4) holds. As the inclusion $W \cdot \lambda \subseteq X(\lambda)$ is tautological, we just need to check that $W \cdot \lambda$ is R-saturated. Let $w \cdot \lambda$ be any element of $W \cdot \lambda$. We need to show that for every α and every integer i between 0 and $(w \cdot \lambda, \alpha^\vee) = \left(\lambda, \left(w^{-1}\alpha\right)^\vee\right) \in \{-1, 0, 1\}$ we have $w\lambda - i\alpha \in W \cdot \lambda$. But this is clear: if $(w \cdot \lambda, \alpha^\vee) = 0$ there is nothing to prove, and if $(w \cdot \lambda, \alpha^\vee) = \pm 1$, then $s_\alpha(w \cdot \lambda) = w \cdot \lambda \mp \alpha$, as required. □

Definition 2.4.4. A weight λ is said to be **minuscule** if it satisfies the above equivalent conditions.

Lemma 2.4.5. *Assume further that $\lambda \in P_{++}(R) \setminus \{0\}$. Let*

$$H = \sum_{j=1}^{l} n_j \alpha_j^\vee$$

be the longest root of $R^\vee$, and let J be the set $\{j | n_j = 1\}$. Then the following are equivalent:

(1) λ is minuscule;

(2) $\lambda(H) = 1$;

(3) there exists $j \in J$ such that $\lambda = \omega_j$.

Proof. Write $\lambda = \sum_{i=1}^{l} u_i\omega_i$, where each u_i is a non-negative integer. Then

$$(\lambda, H) = \left(\sum_{k=1}^{l} u_k\omega_k, \sum_{i=1}^{l} n_i\alpha_i^\vee\right) = \sum_{i,k=1}^{l} n_i u_k \left(\omega_k, \alpha_i^\vee\right) = \sum_{i=1}^{l} n_i u_i$$

equals one if and only if exactly one u_k is 1 (while all the others are zero) and the corresponding n_k is 1: (2) and (3) are then clearly equivalent to each other.

On the other hand, as λ is positive, we get

$$(\lambda, H) = \sup_{h\in R^+} \lambda(h^\vee),$$

so (λ, H) (which is clearly non-zero) equals 1 if and only if all the scalar products $(\lambda, h^\vee)$ are at most 1, that is, if and only if λ is minuscule (cf. condition 4 of the previous Proposition). □

Combining the classification of simple Lie algebras with the characterization (6) of the previous Lemma, it is possible to show the following result (cf. Chapter 8, Section 3 of [Bou08] and Tables 1 and 2, ibid.)

Theorem 2.4.6. *The following table lists all the minuscule weights (along with other useful informations) for the simple Lie algebras. The 'duality properties' column contains 1 if the representation is orthogonal, -1 if it is symplectic, and 0 if it is not self-dual.*

Root system	**Minuscule weight**	**Dimension**	**Duality properties**
$A_l\,(l \geq 1)$	$\omega_r, 1 \leq r \leq l$	$\binom{l+1}{r}$	$(-1)^r$, if $r = \frac{l+1}{2}$ 0, if $r \neq \frac{l+1}{2}$
$B_l\,(l \geq 2)$	ω_l	2^l	$+1$, if $l \equiv 3, 0 \pmod 4$ -1, if $l \equiv 1, 2 \pmod 4$
$C_l\,(l \geq 2)$	ω_1	$2l$	-1
$D_l\,(l \geq 3)$	ω_1	$2l$	$+1$
	ω_{l-1}, ω_l	2^{l-1}	$+1$, if $l \equiv 0 \pmod 4$ -1, if $l \equiv 2 \pmod 4$ 0, if $l \equiv 1 \pmod 2$
E_6	ω_1	27	0
	ω_6	27	0
E_7	ω_7	56	-1

2.4.1 A useful characterization of minuscule weights

In this section we introduce a useful alternative characterization of minuscule weights, due to Zarhin (Section 1.1 of [Zar85]). Let $\lambda = \sum_{i=1}^{l} c_i\alpha_i = \sum_{j=1}^{l} m_j\omega_j$ be a dominant weight. By definition, m_j is a non-negative integer for each j. A simple but useful fact is the following:

Lemma 2.4.7. *c_i is a non-negative rational number for every $i = 1, \dots, l$.*

We shall need a general fact about Euclidean spaces:

Lemma 2.4.8. *Let V be a Euclidean space and let $(\cdot,\cdot)$ denote its positive-definite scalar product. Suppose $v_1, \dots, v_n$ is an obtuse basis of V, that is $(v_i, v_j) \leq 0$ for every pair of different indices i, j.*

Then the dual basis $v_i^\vee$ of v_i with respect to $(\cdot,\cdot)$ is acute, i.e. it verifies $(v_i, v_j) \geq 0$ for every pair of indices.

Proof. We can identify V with $\mathbb{R}^n$ endowed with the standard scalar product. We then proceed by induction on n, the case $n = 2$ being trivial; also, by the symmetry of the problem, it is enough to show $(v_1^\vee, v_2^\vee) \geq 0$. Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{R}^n$. Let now A be any orthogonal transformation. Then

$$(Av_i^\vee, Av_j) = (v_i^\vee, v_j) = \delta_{ij} \quad \forall i, j = 1, \dots, n,$$

so the dual basis of $Av_1, \dots, Av_n$ is $Av_1^\vee, \dots, Av_n^\vee$.

Furthermore, $(Av_i^\vee, Av_j^\vee) = (v_i^\vee, v_j^\vee)$, so the base $(v_i^\vee)_{i=1,\dots,n}$ is acute if and only if $(Av_i^\vee)_{i=1,\dots,n}$ is. We can therefore apply an orthogonal A such that $Av_n = e_n$ and suppose without loss of generality that $v_n = e_n$.

Let π be the (orthogonal) projection

$$\begin{array}{cccc} \pi: & \mathbb{R}^n & \to & \mathbb{R}^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_n = 0\} \\ & (x_1, \dots, x_n) & \mapsto & (x_1, \dots, x_{n-1}, 0). \end{array}$$

For a vector $v \in \mathbb{R}^n$ let $(v^1, \dots, v^n)$ be its coordinates in the standard basis. For every $i < n$, the hypothesis $(v_i, v_n) \leq 0$ implies $v_i^n \leq 0$; on the other hand, for every pair of distinct indices $i, j < n$ we have $(\pi(v_i), \pi(v_j)) = \sum_{i=1}^{n-1} v_i^i v_j^i = (v_i, v_j) - v_i^n v_j^n \leq 0$, since $(v_i, v_j) \leq 0$ by hypothesis and $v_i^n v_j^n$ is non-negative (since both factors are non-positive). It follows that $w_1 = \pi(v_1), \dots, w_{n-1} = \pi(v_{n-1})$ is an obtuse basis of $\mathbb{R}^{n-1}$. Let $w_1^\vee, \dots, w_{n-1}^\vee$ be the dual basis of $\pi(v_1), \dots, \pi(v_{n-1})$ with respect to the scalar product on $\mathbb{R}^{n-1}$.

For any index $i < n$ the last coordinate of $w_i^\vee$ is zero by definition, so $(w_j^\vee, e_n) = 0$ for every $j < n$, and for any two indices $j, k < n$ we can write

$$(w_j^\vee, v_k) = (w_j^\vee, \pi(v_k)) + w_j^n v_k^n = (w_j^\vee, \pi(v_k)) = \left(w_j^\vee, w_k\right) = \delta_{jk}.$$

By uniqueness of the dual basis, this means that the vectors $\left\{w_j^\vee\right\}_{j=1,\dots,n-1}$ are the first $n-1$ elements of the dual basis of $v_1, \dots, v_n$, i.e. $v_i^\vee = w_i^\vee$ for every $i = 1, \dots, n-1$. The inductive hypothesis then implies

$$\left(v_1^\vee, v_2^\vee\right) = \left(w_1^\vee, w_2^\vee\right) \geq 0,$$

and we are done. □

Proof (Lemma 2.4.7). From Remark 2.1.16 we know that each ω_j is a linear combination with rational coefficients of $\alpha_1, \dots, \alpha_n$, so the c_i's are rational.

It is easy to check that for $\alpha_i, \alpha_j \in \Delta$ we have $(\alpha_i, \alpha_j) \leq 0$: suppose by contradiction $(\alpha_i, \alpha_j) > 0$. As clearly $\alpha_j \neq \pm\alpha_i$, Lemma 2.1.10 implies that $\alpha_i - \alpha_j$ is a root, so it admits an expression $\alpha_i - \alpha_j = \sum_{k=1}^{l} n_k \alpha_k$ where each n_k a non-negative or non-positive integer. As this would give two different representations of α_i (namely α_i itself and $(\alpha_i - \alpha_j) + \alpha_j$), the contradiction shows that we must have $(\alpha_i, \alpha_j) \leq 0$.

We then have $\left(\alpha_i^\vee, \alpha_j^\vee\right) \leq 0$, and thanks to the previous Lemma we know that the dual basis $\{\omega_i\}_{i=1,\dots,n}$ verifies $(\omega_i, \omega_j) \geq 0$.

It follows that for each $j = 1, \dots, n$ the scalar product

$$(\lambda, \omega_j) = \sum_{i=1}^{l} m_i \left(\omega_i, \omega_j\right)$$

is non-negative, hence

$$0 \leq (\lambda, \omega_j) = \left(\sum_{i=1}^{l} c_i \frac{(\alpha_i, \alpha_i)}{2} \alpha_i^\vee, \omega_j\right) = c_j \frac{(\alpha_j, \alpha_j)}{2}$$

and since $(\alpha_j, \alpha_j) > 0$ we deduce $c_j \geq 0$, as claimed. □

Let λ' be $-w_0(\lambda)$, where w_0 is the opposition involution, and write $\lambda' = \sum_{i=1}^{l} c_i' \alpha_i$. Define $l(\lambda) := \min\left\{c_i + c_i' | i = 1, \cdots, l\right\}$.

Remark 2.4.9. As all the coefficients c_i and c_i' are non-negative, $l(\lambda)$ is positive for every non-trivial dominant weight λ.

Lemma 2.4.10. *l is an integer-valued function.*

Proof. Pick any $\alpha \in R$ and let $w \in W$ be the reflection through the hyperplane orthogonal to α. Write $\mu = \sum_{i=1}^{l} a_i \omega_i$ with $a_i \in \mathbb{Z}$ for a weight. Then

$$w(\mu) = \mu - 2\frac{(\alpha, \mu)}{(\alpha, \alpha)}\alpha = \mu - (\mu, \alpha^\vee)\alpha,$$

and $(\mu, \alpha^\vee)$ is an integer by definition of weight. Moreover, α itself is a weight, so $\mu - (\mu, \alpha^\vee)\alpha$ is again a weight; furthermore, α is a $\mathbb{Z}$-linear combination of simple roots. A simple induction (on the number of symmetries involved) then shows that for any weight μ and any element $w \in W$ we have

$$w(\mu) = \mu - \sum_{i=1}^{l} q_i \alpha_i, \quad q_i \in \mathbb{Z}\ \forall i.$$

Applying this to $w = w_0$ and $\mu = \lambda$ we get

$$\sum_{i=1}^{l} c_i' \alpha_i = \lambda' = -w_0(\lambda) = -\left(\lambda - \sum_{i=1}^{l} q_i \alpha_i\right) = \sum_{i=1}^{l} q_i \alpha_i - \sum_{i=1}^{l} c_i \alpha_i,$$

so $c_i + c_i' = q_i \in \mathbb{Z}$, as we wanted to show. □

The following proposition can be useful in order to bound from below the value of l:

Proposition 2.4.11. *Let $\tilde{\beta}$ be the longest root of $R^\vee$. Then for every dominant weight λ we have $l(\lambda) \geq (\lambda, \tilde{\beta})$*

Proof. Let $\gamma = \sum_{i=1}^{l} e_i \alpha_i$ be a positive root (so each e_i is a non-negative integer). Then the associated reflection $s_\gamma \in W$ acts as

$$w_\gamma(\lambda) = \lambda - (\lambda, \gamma^\vee)\gamma = \lambda - \sum_{i=1}^{l} (\lambda, \gamma^\vee) e_i \alpha_i$$

It is known that $w_0(\lambda) \prec w_\gamma(\lambda)$, so

$$\lambda - \sum_{i=1}^{l} (\lambda, \gamma^\vee) e_i \alpha_i \succ w_0(\lambda) \Rightarrow \lambda + \lambda' \succ \sum_{i=1}^{l} (\lambda, \gamma^\vee) e_i \alpha_i,$$

that in turn implies $c_i + c_i' \geq (\lambda, \gamma^\vee) e_i$. For $\gamma = \tilde{\beta}^\vee$ we have $e_i \geq 1$ for every index i, so $c_i + c_i' \geq (\lambda, \tilde{\beta})$ holds for every i and the claim follows. □

The importance of l lies in the following property (1.1.2.0 of [Zar85]):

Proposition 2.4.12. *Let λ be a dominant weight. Then $l(\lambda) = 1$ if and only if λ is minuscule and (V, R) is of classical type.*

Another equivalent characterization of $l(\lambda)$ is given in the following proposition:

Proposition 2.4.13. *Let $\mathfrak{g}$ be a simple Lie algebra, (V, R) the associated root system, λ a dominant weight of R and X the small saturated set containing λ. Then*

1. *for every non-trivial group morphism $\varphi : P(R) \to (\mathbb{Q}, +)$ we have $|\varphi(X)| - 1 \geq l(\lambda)$;*
2. *there exists a group morphism $\varphi : P(R) \to (\mathbb{Q}, +)$ such that $|\varphi(X)| = l(\lambda) + 1$.*

Proof. If $\lambda = 0$ the claim is clear, so we can assume that $\lambda \neq 0$ and $\mathfrak{g}$ admits an irreducible, faithful representation $V(\lambda)$ having λ as its highest weight.

Let $\varphi : P(R) \to \mathbb{Q}$ be a non-trivial group morphism. $P(R)$ is a lattice in V (in particular, it has maximal rank), so identifying $V \cong V^*$ through the given inner product we find that φ is given by $\varphi(\mu) = \alpha \cdot (\mu, \gamma^\vee)$ for a certain $\gamma \in P(R)$ and a certain constant $\alpha \in \mathbb{Q}$.

Replacing φ with $\mu \mapsto (\mu, \gamma^\vee)$ does not change the cardinalities of the involved sets, so we can assume that φ is indeed given by $\mu \mapsto (\mu, \gamma^\vee)$. As X is stable under the Weyl group, and since the action of W on V is through isometries, we can replace $\gamma^\vee$ with any W-conjugate. In particular, this means that we can assume $\gamma \in P_{++}(R)$, since every W-orbit meets $P_{++}(R)$.

We now remark that X is exactly the set of weights of $V(\lambda)$ by the structure theorem for these representations (essentially a consequence of the Poincaré-Birkhoff-Witt theorem), so we know that for every $\mu \in X$ there exists a sequence $\lambda_0 = \lambda, \lambda_1, \cdots, \lambda_n = \mu$ such that $\lambda_i - \lambda_{i+1}$ is a simple root for every $i = 0, \cdots, n-1$.

Let's now write $\lambda = \sum_{i=1}^{l} c_i \alpha_i$ and $\mu = w_0(\lambda) = \lambda - \sum_{i=1}^{l} q_i \alpha_i$, where we know the q_i's to be integers with the property that $l(\lambda) = \inf_i q_i$ (cf. the proof of Lemma 2.4.10). In particular, every q_i is greater than or equal to $l(\lambda)$, and given any $\gamma \in P(R)$ there exists $i \in \{1, \ldots, l\}$ such that $(\alpha_i, \gamma^\vee) \neq 0$ (since R spans V). On the other hand, for any index $j \in \{1, \ldots, l\}$ we see that in the sequence $\lambda_0 - \lambda_1, \cdots, \lambda_{n-1} - \lambda_n$ there are exactly q_j terms equal to α_j (by uniqueness of the representation in terms of a base).

Putting everything together we find that

$$\varphi(X) \supseteq \varphi\left(\{\lambda_0, \cdots, \lambda_n\}\right) =$$

$$= \left\{ \varphi(\lambda_n), \varphi(\lambda_n) + \varphi(\lambda_{n-1} - \lambda_n), \cdots, \varphi(\lambda_n) + \sum_{i=0}^{n-1} \varphi(\lambda_i - \lambda_{i+1}) \right\},$$

and for any fixed index j this last set has cardinality at least $q_j + 1$: all the summands $\varphi(\lambda_i - \lambda_{i+1})$ are non-negative, since they are of the form (simple root, $\gamma^\vee$) with γ dominant, and if $\lambda_k - \lambda_{k+1} = \alpha_j$, then $\varphi(\lambda_k - \lambda_{k-1}) >$

0, which implies that for at least q_j values of the integer k we have $\varphi(\lambda_k) \neq \varphi(\lambda_{k+1})$.

Let us now choose an index r realizing the lower bound, i.e. an r such that $l(\lambda) = q_r$. Define φ to be the homomorphism $\mu \mapsto (\mu, \omega_r)$. The above calculation then shows that

$$|\varphi(X)| = 1 + |\{i : \lambda_i - \lambda_{i+1} = \alpha_r\}| = 1 + q_r,$$

which proves (2). □

2.4.2 Computations for A_l

To show that the whole theory is in fact very explicit we now determine the minuscule weights for simple root systems of type A_l and check that λ is minuscule if and only if $l(\lambda) = 1$.

We consider the following embedding of the root system A_l: let V be the hyperplane $\{x_1 + \cdots + x_{l+1} = 0\}$ of $\mathbb{R}^{l+1}$, and take as R the set of vectors of V with integral coordinates and squared norm 2. As a base we take $\Delta = \{a_1 = e_1 - e_2, \cdots, a_l = e_l - e_{l+1}\}$, so that the positive roots are those of the form $e_i - e_j$ for $i < j$. The Cartan matrix associated to this choice is

$$(C)_{kl} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

as it is easy to see:

$$C_{ij} = 2\frac{(a_i, a_j)}{(a_i, a_i)} = (a_i, a_j) = (e_i - e_{i+1}, e_j - e_{j+1}) = \begin{cases} 2, \text{ if } i = j \\ -1, \text{ if } |i - j| = 1 \\ 0, \text{ otherwise} \end{cases}$$

The Weyl group is generated by the reflections σ_i along the vectors $e_i - e_{i+1}$, and σ_i acts on a vector $(x_1, \cdots, x_{l+1})$ as

$$(x_1, \cdots, x_{l+1}) \mapsto (x_1, \cdots, x_{l+1}) - 2\frac{((x_1, \cdots, x_{l+1}), e_i - e_{i+1})}{(e_i - e_{i+1}, e_i - e_{i+1})}(e_i - e_{i+1}) =$$

$$= (x_1, \cdots, x_{l+1}) - (x_i - x_{i+1})(e_i - e_{i+1}) = (x_1, \cdots, x_{i+1}, x_i, \cdots, x_{l+1}),$$

i.e. by transposing the i^{th} and $(i+1)^{th}$ coordinate. Since transpositions generate the symmetric group, we deduce that W is isomorphic to S_{l+1}, the full permutation group on $l+1$ elements. The Weyl chambers correspond to the choice of a linear orders on the coordinates (for example, in the case $l=2$ the six Weyl chambers are given by $\{(x_1,x_2,x_3) : x_1 > x_2 > x_3\}$, $\{(x_1,x_2,x_3) : x_1 > x_3 > x_2\}$, and so on), and our base is associated to the chamber given by $x_1 > x_2 > \cdots > x_{l+1}$. The opposite involution carries it to the chamber given by $-x_1 > -x_2 > \cdots > -x_{l+1}$, i.e. $x_1 < x_2 < \cdots < x_{l+1}$. By uniqueness, and since reversing the order of the coordinates (i.e. $(x_1, x_2, \cdots, x_l, x_{l+1}) \mapsto (x_{l+1}, x_l, \cdots, x_2, x_1)$) works, this must be the opposition involution w_0.

On the roots, w_0 acts as $w_0(\alpha_i) = w_0(e_i - e_{i+1}) = e_{l+2-i} - e_{l+1-i} = -\alpha_{l+1-i}$.

Finally, the longest root of $R^\vee = R$ is $\tilde{\beta} := e_1 - e_{l+1} = \sum_{j=1}^l \alpha_j = \sum_{j=1}^l \alpha_j^\vee$, so thanks to (6) of Lemma 2.4.5 we know that the positive minuscule weights are exactly the dominant weights ω_j.

We now want to check that, for a dominant λ, $l(\lambda) = 1$ if and only if λ is one among the ω_j's. Write

$$\lambda = \sum_{i=1}^l c_i \alpha_i = \sum_{i=1}^l m_i \omega_i.$$

If $l(\lambda) = 1$, Proposition 2.4.11 yields $\left(\lambda, \tilde{\beta}\right) = 1$, so

$$1 = \left(\sum_{i=1}^l m_i \omega_i, \sum_{j=1}^l \alpha_j \right) = \sum_i m_i.$$

As all the m_i's are non-negative integers, it follows that exactly one of them equals one while all the others vanish, so λ is one among the ω_j's, as we wanted to show.

On the other hand, from the above it is easy to compute

$$-w_0(\lambda) = -\sum_{i=1}^l c_i \left(-\alpha_{l+1-i}\right) = \sum_{i=1}^l c_{l+i-1} \alpha_i,$$

so

$$l(\lambda) = \min_{i=1,\cdots,l} \left(c_i + c_{l+1-i}\right).$$

We want to express the coefficients c_i in terms of the integers m_i. To this end, simply note that

$$m_j = \left(\sum_{i=1}^l m_i \omega_i, \alpha_j^\vee \right) = \left(\lambda, \alpha_j^\vee\right) = \left(\sum_{i=1}^l c_i \alpha_i, \alpha_j^\vee \right) = \sum_{i=1}^l c_i \langle \alpha_j, \alpha_i \rangle = \sum_{i=1}^l C_{ji} c_i,$$

so, writing D for the inverse of the Cartan matrix (or rather, of its transpose - but in this case C is symmetric, so the two coincide), we have

$$c_i = \sum_{j=1}^{l} D_{ij} m_j.$$

It is easy to check that

$$D_{ij} = \begin{cases} \dfrac{(l+1-i)j}{l+1}, \text{ if } i \geq j \\ \\ \dfrac{(l+1-j)i}{l+1}, \text{ if } j \geq i \end{cases},$$

so - given any choice $(m_1, \cdots, m_l)$ for the coefficients of the fundamental weights - we get

$$c_i + c_{l+1-i} = \sum_{j=1}^{l} \left(D_{i,j} m_j + D_{l+1-i,j} m_j \right).$$

Suppose now that λ is minuscule, i.e. that there exists exactly one index $j \in \{1, \cdots, l\}$ such that $m_j = 1$, while all the others are zero. Using $D_{1,k} = \dfrac{l+1-k}{l+1}$ and $D_{l,k} = \dfrac{k}{l+1}$, $i = 1$ gives

$$c_1 + c_l = \sum_{k=1}^{l} \left(\frac{l+1-k}{l+1} + \frac{k}{l+1} \right) m_j = 1.$$

As l is integer-valued and strictly positive (see Remark 2.4.9), $l(\lambda) \leq c_1 + c_l$ forces $l(\lambda) = 1$, as claimed. This completes the verification of the equivalence $l(\lambda) = 1 \Longleftrightarrow \lambda$ is minuscule for the root systems of type A_l.

CHAPTER 3

Mumford-Tate and Hodge groups

Following the unpublished course notes by B. Moonen ([Mooa], [Moob]) we now introduce the main objects we will have to deal with and prove a few of their most basic properties.

The first section is dedicated to the definition of the Mumford-Tate and Hodge groups and their most immediate properties. In 3.2 we introduce the notion of a polarization for an abstract Hodge structure and derive important properties enjoyed by polarizable structures.

We also clarify the relation between the abstract notion and its geometric counterpart, which is sometimes left implicit in the literature, showing in the process that Hodge structures coming from Abelian varieties are polarizable.

In the following section we then investigate the Hodge group of a product of varieties, and we conclude the chapter by analyzing, in 3.4, the interactions among polarizations, the endomorphism algebra and the Hodge group. This will also lead to introducing the so-called Lefschetz group.

3.1 Definition and basic properties

Throught this section V is a fixed $\mathbb{Q}$-Hodge structure of weight m and

$$h : \mathbb{S} \to GL(V_{\mathbb{R}})$$

is the associated morphism of algebraic groups.

We introduce here one of the true protagonists of this work:

Definition 3.1.1. The **Mumford-Tate group** of V, denoted $MT(V)$, is the smallest algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$ such that h factors through $MT(V)_{\mathbb{R}}$.

It is an exercise in [Mooa] to show that it is possible to give the following alternative characterization of $MT(V)$:

Proposition 3.1.2. *Let $A(V)$ be the smallest algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$ such that $h_{\mathbb{C}} \circ \mu : \mathbb{G}_{m,\mathbb{C}} \to GL(V_{\mathbb{C}})$ factors through $A_{\mathbb{C}}$. Then $A(V) = MT(V)$.*

Proof. We have an immediate inclusion $A(V) \subset MT(V)$: indeed, the very definition of $MT(V)$ implies that h factors through $MT(V)_{\mathbb{R}}$, so by extending scalars to $\mathbb{C}$ we find that $h_{\mathbb{C}}$ factors through $MT(V)_{\mathbb{C}}$; a fortiori, $h_{\mathbb{C}} \circ \mu$ factors through $MT(V)_{\mathbb{C}}$, and since A is the smallest subgroup with this property we must have $A(V) \subset MT(V)$.

On the other hand, A is defined over $\mathbb{Q}$, so it is stable under the action of $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$; in particular, its image in $GL(V_{\mathbb{C}})$ is stable under τ, the complex conjugation. Write $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ for the Hodge decomposition of V, where $V^{p,q}$ is the space of vectors on which the action of z is given by $z^{-p}\overline{z}^{-q}$. By definition, $A_{\mathbb{C}}$ contains the image of $h \circ \mu$, that is, it contains all the operators of the form

$$h_{\mathbb{C}} \circ \mu(z) = \bigoplus_{p+q=n} z^{-p} \operatorname{id}_{V^{p,q}},$$

since by definition of μ we have $z \circ \mu = id$ (resp. $\bar{z} \circ \mu = 1$).

Writing down the action of τ we find that $A_{\mathbb{C}}$ contains the elements

$$\tau \cdot \left(\bigoplus_{p+q=n} z^{-p} \operatorname{id}_{V^{p,q}} \right) = \tau \circ \left(\bigoplus_{p+q=n} z^{-p} \operatorname{id}_{V^{p,q}} \right) \circ \tau,$$

and since τ exchanges $V^{p,q}$ and $V^{q,p}$ this is just

$$\bigoplus_{p+q=n} \overline{z}^{-q} \operatorname{id}_{V^{p,q}} .$$

Finally, as $A_{\mathbb{C}}$ is closed under composition, it also contains

$$\bigoplus_{p+q=n} z_1^{-p} \overline{z_2}^{-q} \operatorname{id}_{V^{p,q}}$$

for every choice of $z_1, z_2 \in \mathbb{C}^* \times \mathbb{C}^*$. It follows immediately from the identifications given in Remark 1.1.7 that this is precisely $h_{\mathbb{C}}(\mathbb{S})$, so $h_{\mathbb{C}}$ factors through $A_{\mathbb{C}}$, h factors through $A_{\mathbb{R}}$ and therefore $A(V) \supset MT(V)$. $\square$

Definition 3.1.3. The **unit circle group** $\mathbb{U}_1 \subset \mathbb{S}$ is the kernel of the norm character. Its real points correspond to the unit circle S^1.

We can now finally introduce what is probably the most important notion in the theory:

Definition 3.1.4. The **Hodge group** $Hg(V)$ is the smallest algebraic subgroup of $GL(V)$ defined over $\mathbb{Q}$ such that $h|_{\mathbb{U}_1}$ factors through $Hg(V)_{\mathbb{R}}$.

Proposition 3.1.5. *For any Hodge structure V, both $MT(V)$ and $Hg(V)$ are connected.*

Proof. $\mathbb{S}$ is connected, so the morphism $h : \mathbb{S} \to GL(V_{\mathbb{R}})$ defining the Hodge structure V factors through a group G if and only if it factors through G^0. By minimality $MT(V)$ must then equal $MT(V)^0$, so it is connected. Same proof for the Hodge group. □

Proposition 3.1.6. *For $m = 0$ the Hodge and Mumford-Tate groups coincide, while for $m \neq 0$ $MT(V)$ is the almost-direct product of $\mathbb{G}_{m,\mathbb{Q}}$ and $Hg(V)$.*

Proof. It is apparent from the definitions that the Hodge group of V is a subgroup of $MT(V)$. Let $m = 0$ and write $z = |z|e^{i\theta}$ for the polar form of complex numbers. Noticing that $h|_{\mathbb{R}^*}$ is trivial, we find that h factors through a subgroup G of $GL(V)$ if and only if $h|_{\mathbb{U}_1}$ does, since $h(z) = h(z/|z|)$ and $z/|z|$ is on S^1. The definitions of $MT(V)$ and $Hg(V)$ then immediately imply the desired equality.

Suppose from now on that $m \neq 0$. As a first step, we compute the determinant of $h(z) \in GL(V_{\mathbb{R}})$ for any $z \in \mathbb{C}^* = \mathbb{S}(\mathbb{R})$. Since $\det(h(z))$ is invariant under extension of scalars, we may as well compute the determinant of $h(z)_{\mathbb{C}} \in GL_{\mathbb{C}}(V)$. In view of the convention set up in Remark 1.1.8 we know that $h(z)_{\mathbb{C}}$ acts as multiplication by $z^{-p}\bar{z}^{-q}$ on $V^{p,q}$, so

$$\begin{aligned}\det(h(z)) = \det(h(z)_{\mathbb{C}}) &= \prod_{p+q=m} \det\left(z^{-p}\bar{z}^{-q}\,\mathrm{Id}_{V^{p,q}}\right) \\ &= \prod_{p+q=m} \left(z^{-p}\bar{z}^{-q}\right)^{\dim(V^{p,q})}.\end{aligned}$$

Exploiting the symmetry $\overline{V^{p,q}} = V^{q,p}$ we can rewrite the above expression as

$$\begin{aligned}\det(h(z)) &= \prod_{p+q=m} \left(z^{-p}\bar{z}^{-q}\right)^{\frac{1}{2}\dim(V^{p,q})} \left(z^{-q}\bar{z}^{-p}\right)^{\frac{1}{2}\dim(V^{q,p})} \\ &= \prod_{p+q=m} Nm(z)^{(-p-q)\cdot\frac{1}{2}\dim(V^{p,q})} \\ &= Nm(z)^{-\frac{1}{2}m\cdot\dim\left(\bigoplus_{p+q=m} V^{p,q}\right)} \\ &= Nm(z)^{-\frac{1}{2}m\dim(V)}.\end{aligned}$$

In particular, as $\mathbb{U}_1$ is the kernel of Nm, we see that $Hg(V) \subset SL(V)$. As $h \circ w(a) = a^{-m}\,\mathrm{Id}_V$ and m is nonzero, the image of h contains all of the real homotheties, therefore $MT(V) \supset \mathbb{G}_{m,\mathbb{Q}}$. We can then compute the product of $\mathbb{G}_{m,\mathbb{Q}}$ and $Hg(V)$ inside $MT(V)$.

The intersection of these two subgroups is clearly finite, since (even over the algebraic closure $\mathbb{C}$) there is only a finite number of points in $SL(\mathbb{C}) \cap \mathbb{G}_{m,\mathbb{C}}(\mathbb{C})$, correseponding to the roots of unity of order dividing $\dim(V)$. Moreover, both subgroups are normal, since $\mathbb{G}_{m,\mathbb{Q}}$ is contained in the center and Hg can be identified with the kernel of the determinant morphism (restricted to MT). On the other hand, these two subgroups generate $MT(V)$: as we have already remarked, writing a complex number z as $|z|u$ with $u \in \mathbb{U}_1(\mathbb{R})$ gives $h(z) = h(|z|)h(u)$ as the product of a real homothety and an element of $Hg(V)_{\mathbb{R}}$, so we finally get $MT(V) = \mathbb{G}_{m,\mathbb{Q}} \cdot Hg(V)$. □

Proposition 3.1.7. *Let V be a Hodge structure and $\nu = \{(a_i, b_i)\}_{i=1,\cdots,n}$ be a collection of pairs of non-negative integers. The vector space*

$$V^{\nu} = \sum_{i=1}^{n} V^{\otimes a_i} \otimes \left(V^{\vee}\right)^{\otimes b_i}$$

inherits a Hodge structure from V, and the tautological representation of $MT(V)$ induces an action of $MT(V)$ on V^{ν}.

Let W be a vector subspace of V^{ν}. Then W is a sub-Hodge structure if and only if W is invariant under the action of $MT(V)$, and an element $t \in V^{\nu}$ is a Hodge class if and only if it is invariant under the action of $MT(V)$.

Proof. Let $H < GL(V^{\nu})$ be the stabilizer of the subspace W in V^{ν}, i.e. the algebraic subgroup given on $\mathbb{Q}$-algebras A by

$$H(A) = \left\{x \in GL\left(V^{\nu} \otimes A\right) | x(W \otimes A) \subseteq W \otimes A\right\}.$$

Note that the action of x on W is the one given by representing $GL(V)$ in $GL(V^{\nu})$. As W is a rational subspace, H is clearly defined over $\mathbb{Q}$; consider now the morphism $h : \mathbb{S} \to GL(V_{\mathbb{R}})$. By definition, W is a sub-Hodge structure if and only if it a subrepresentation for the action of $\mathbb{S}$, if and only if h factors through $H_{\mathbb{R}}$, and this happens exactly if H contains the Mumford-Tate group of V, namely if and only if $MT(V)$ stabilizes W.

For the second assertion, given any ν we can take $\nu' = ((0,0), \nu)$ to get $V^{\nu'} \cong \mathbb{Q}(0) \oplus V^{\nu}$, and t is a Hodge class in V^{ν} if and only if the subspace $\mathbb{Q} \cdot (1, t) \subseteq \mathbb{Q}(0) \oplus V^{\nu}$ is a sub-Hodge structure. The result then follows from the first part. □

The following result, often quoted as *Riemann's Theorem* (see, for example, Corollary 6.9 in [Mil05]), is the reason why Hodge structures are so useful in studying Abelian varieties. We state it here in provisional form, and postpone the full result until after the definition of a polarizable Hodge structure (see Theorem 3.2.6 below).

Theorem 3.1.8. *The functor $A \mapsto H_1(A, \mathbb{Q})$ induces a full embedding of the category of Abelian varieties over $\mathbb{C}$, up to isogeny, into the category of $\mathbb{Q}$-Hodge structures of type $(-1,0),(0,-1)$.*

If A is an Abelian variety, with a little abuse of notation we will speak of the Mumford-Tate (resp. Hodge) group of A, meaning the corresponding group of the Hodge structure $H_1(A, \mathbb{Q})$.

Proposition 3.1.9. *Let A be a simple Abelian variety, D its endomorphism algebra, $V = H_1(A, \mathbb{Q})$, M its Mumford-Tate group. Then*

$$D \cong (\mathrm{End}(V))^M = (\mathrm{End}(V))^{Hg(A)}$$

Proof. The last equality follows immediately, since we already know (Proposition 3.1.6) that in this case $MT(A) = Hg(A) \cdot \mathbb{G}_{m,\mathbb{Q}}$, and clearly $\mathbb{G}_{m,\mathbb{Q}}$ (acting through multiples of the identity) commutes with every automorphism of V.

In view of the above theorem, the endomorphism algebra D of A is isomorphic to the endomorphism algebra of the Hodge structure $V := H_1(A, \mathbb{Q})$. Let $h : \mathbb{S} \to GL(V_\mathbb{R})$ be the morphism describing the Hodge structure V. A (Hodge) endomorphism φ of V is a $h(\mathbb{S})$-equivariant map, that is, a linear transformation such that

$$\varphi_\mathbb{R}(h(z)v) = h(z)\varphi_\mathbb{R}(v) \quad \forall v \in V_\mathbb{R}, \forall z \in \mathbb{S}(\mathbb{R}).$$

Replacing v by $h(z)^{-1}v$ we see that φ is a endomorphism of the Hodge structure V if and only if

$$h(z) \circ \varphi \circ h(z)^{-1} = \varphi \quad \forall z \in \mathbb{S}(\mathbb{R}).$$

But this is exactly the definition of the action of $\mathbb{S}$ on $\mathrm{End}(V)$, so φ is an endomorphism of V as a Hodge structure if and only if it is a Hodge class in $\mathrm{End}(V)$. Thanks to Proposition 3.1.7 we know that Hodge classes equal the fixed points for the action of $MT(V)$, so we finally get the desired equality. □

There is also a particularly useful characterization of CM varieties in terms of their Mumford-Tate groups:

Proposition 3.1.10. *Let A be a simple abelian variety over $\mathbb{C}$ with Mumford-Tate group M. Then A has complex multiplication if and only if M is a torus.*

Proof. Suppose first that A has complex multiplication. Let g be the dimension of A. Then by definition $D := End^0(A)$ is a field of degree $2g$ over $\mathbb{Q}$, and since $V := H_1(A, \mathbb{Q})$ is a D-module of dimension $2g$ over $\mathbb{Q}$ we find that

V is a free D-module of rank 1. We can then identify $V \cong D$ (with its natural action of D by left multiplication).

Let $\varphi \in M$. Since the actions of M and D on $V = D$ commute by Proposition 3.1.9, we get

$$\varphi(d) = \varphi(d \cdot 1) = d\varphi(1) \quad \forall d \in D,$$

so φ can be identified to multiplication by $\varphi(1)$ (D being commutative, there is no need to distinguish left and right multiplication here). It follows that M is a subgroup of $\mathrm{Res}_{D/\mathbb{Q}}(\mathbb{G}_m)$, and being connected (Proposition 3.1.5) and reductive it is a torus.

On the other hand, suppose M is a torus and let T be any maximal torus in $GL(V)$ containing T. Then $D = \mathrm{End}(V)^M \supseteq \mathrm{End}(V)^T$, and since everything commutes with extending scalars we find

$$D \otimes_{\mathbb{Q}} \mathbb{C} \supseteq \mathrm{End}(V \otimes_{\mathbb{Q}} \mathbb{C})^{T_{\mathbb{C}}}.$$

Since all maximal tori are conjugated over $\mathbb{C}$, we can simply take $T_{\mathbb{C}}$ to be the diagonal torus, in which case $\mathrm{End}(V \otimes_{\mathbb{Q}} \mathbb{C})^{T_{\mathbb{C}}}$ equals the set of diagonal matrices, which has dimension $2g$ over $\mathbb{C}$. Taking dimension then yields

$$\dim_{\mathbb{Q}}(D) = \dim_{\mathbb{C}}(D \otimes_{\mathbb{Q}} \mathbb{C}) \geq \dim_{\mathbb{C}}\left(\mathrm{End}(V \otimes_{\mathbb{Q}} \mathbb{C})^{T_{\mathbb{C}}}\right) = 2g,$$

which by definition means that A has complex multiplication.

□

3.2 Polarizable Hodge structures

An extremely important subclass of Hodge structure is given by the so-called **polarizable** ones. As the name suggests, this notion comes from geometry, but in order to introduce the Hodge counterparts of polarizations we first need to define the Tate structures $\mathbb{Q}(n)$.

Definition 3.2.1. For every integer $n \in \mathbb{Z}$, the **Tate structure** $\mathbb{Q}(n)$ is the vector space $V := \mathbb{Q} \cdot (2\pi i)^n \subset \mathbb{C}$, with Hodge structure purely of type $(-n, -n)$: in other words,

$$V_{\mathbb{C}} \cong \mathbb{C}$$

is declared to be $V^{-n,-n}$. The **n-th Tate twist** of a Hodge structure W is $W(n) := W \otimes_{\mathbb{Q}} \mathbb{Q}(n)$.

Remark 3.2.2. An element $z \in \mathbb{S}(\mathbb{R})$ acts on $V_{\mathbb{C}}$ via the multiplication by $z^n\bar{z}^n = Nm(z)^n$, which is to say that the homomorphism $h : \mathbb{S}(\mathbb{R}) \to GL(V_{\mathbb{R}})$ is simply Nm^n. It is then clear from the definitions that the Tate structure $\mathbb{Q}(n)$ is of pure weight $-2n$.

Definition 3.2.3. Let V be a Hodge structure of weight n with Weil operator C. A **polarization** of V is a morphism of Hodge structures

$$\varphi : V \otimes V \to \mathbb{Q}(-n)$$

such that the bilinear form (on the *real* vector space $V_{\mathbb{R}}$)

$$\begin{array}{ccc} V_{\mathbb{R}} \times V_{\mathbb{R}} & \to & \mathbb{R} \\ (x, y) & \mapsto & (2\pi i)^n \varphi(Cx \otimes y) \end{array}$$

is symmetric and positive definite.

A pair (V, φ), where V is a Hodge structure and φ is a polarization of V, is called a **polarized Hodge structure**. A Hodge structure admitting at least one polarization is called **polarizable**.

Proposition 3.2.4. *Let φ be a polarization of a structure V of weight n. Then φ is symmetric (resp. skew-symmetric) if n is even (resp. odd).*

Proof. A morphism of Hodge structures commutes with the Weil operator by Remark 1.1.11; moreover, the Weil operator of $\mathbb{Q}(n)$ is trivial, since $i \in \mathbb{S}(\mathbb{R})$ acts as multiplication by $Nm(i)^n = 1$. Furthermore, note that if C is the Weil operator of V, then $C \otimes C$ is the Weil operator of $V \otimes V$.

It follows that

$$\begin{aligned} \varphi(Cx \otimes y) &= \varphi(Cy \otimes x) && \text{(symmetry)} \\ &= C\varphi(Cy \otimes x) && \text{(triviality of } C \text{ on } \mathbb{Q}(n)) \\ &= \varphi(C^2 y \otimes Cx) && (\varphi \text{ commutes with } C) \\ &= \varphi((-1)^n y \otimes Cx) && (C_V^2 = (-1)^n) \\ &= (-1)^n \varphi(y \otimes Cx) && \text{(linearity of } \varphi), \end{aligned}$$

so (as C is an automorphism) $\varphi(x \otimes y) = (-1)^n \varphi(y \otimes c)$, as claimed. □

Remark 3.2.5. As it is well-known, there is also a notion of polarizations for Abelian varieties. This is no coincidence, although the relation between the two objects is not completely apparent.

For the sake of simplicity let's work over $\mathbb{C}$, so that we can write an Abelian variety X as W/Λ, where W is a g-dimensional vector space over $\mathbb{C}$ and Λ is a (full-rank) lattice in W. An equivalent notion of polarization on X is then that a non-degenerate, positive-definite Hermitian form

$$H : W \times W \to \mathbb{C}$$

such that $\mathrm{Im}(H)(\Lambda, \Lambda) \subseteq \mathbb{Z}$.

We now want to explore the relation between this notion, our previous definition and polarizations at the level of Hodge structures.

There is a bijection between such forms H and alternating $\mathbb{R}$-linear forms $E : W \times W \to \mathbb{R}$ such that $E(iu, iv) = E(u, v)$ for all $u, v \in W$, the correspondence being given by $H(u, v) = E(u, iv) + iE(u, v)$. Moreover, H satisfies the above properties if and only if E satisfies

1. E is non-degenerate
2. the symmetric form $(u, v) \mapsto E(u, iv)$ is positive definite
3. E is integer-valued on $\Lambda \times \Lambda$.

The long exact cohomology sequence associated to the exponential exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

gives rise to a map (the first Chern class)

$$c_1 : H^1\left(X, \mathcal{O}_X^*\right) \to H_2\left(X, \mathbb{Z}\right) \cong \operatorname{Hom}\left(\bigwedge^2 \Lambda, \mathbb{Z}\right),$$

whose image consists of alternating forms E 'of type $(1,1)$', i.e. such that $E_{\mathbb{R}}(iu, iv) = E_{\mathbb{R}}(u, v)$. Such E's clearly satisfy (3) above; it is then natural to ask what conditions we ought to impose on a line bundle $\mathcal{L}$ in order for $c_1(\mathcal{L})$ to be positive-definite.

Consider the dual torus $\hat{X}$ of X. It can be described as $\hat{\overline{W}}/\hat{\Lambda}$, where $\hat{\overline{W}}$ is the $\mathbb{C}$-vector space of $\mathbb{C}$-antilinear forms $W \to \mathbb{C}$ and $\hat{\Lambda}$ is the dual lattice

$$\hat{\Lambda} = \left\{ h \in \hat{\overline{W}} | h\left(\Lambda\right) \subset \mathbb{Z} \right\}$$

Let $\mathcal{L}$ be a holomorphic line bundle on X and let H be the Hermitian form on W corresponding to $c_1(\mathcal{L})$. Associated to $\mathcal{L}$ we have a morphism

$$\begin{array}{rccc} \varphi_{\mathcal{L}} : & X & \to & \hat{X} \\ & v & \to & H(v, -), \end{array}$$

and it turns out that the form H is non-degenerate if and only if $\varphi_{\mathcal{L}}$ is an isogeny. Moreover, H is positive definite if and only if $\mathcal{L}$ is ample, i.e. if and only if $\varphi_{\mathcal{L}}$ is a polarization in the geometric sense. Since every polarization has an associated ample line bundle, we see that the data of a polarization is equivalent to the data of H, which is in turn equivalent to giving an E as above, which is exactly a polarization in the sense of Hodge structures.

We can now state the full version of Riemann's Theorem as follows:

Theorem 3.2.6. *The functor $A \mapsto H_1(A, \mathbb{Q})$ is an equivalence between the category of Abelian varieties over $\mathbb{C}$, up to isogeny, and the category of polarizable $\mathbb{Q}$-Hodge structures of type $(-1, 0), (0, -1)$.*

Remark 3.2.7. Let (V, φ) be a polarized Hodge structure and W be a sub-Hodge structure of V. Then W is again polarizable (we can take as polarization the restriction of φ), and the orthogonal complement $W^\perp$ of W with respect to the bilinear form induced by φ is again a sub-Hodge structure. Indeed, $W^\perp$ is a sub-Hodge structure if and only if it is stable under the action of $Hg(V)$, but this is rather clear: the identity

$$\varphi\left(w \otimes hw^\perp\right) = \varphi\left(h^{-1}w \otimes w^\perp\right) = 0 \quad \forall w \in W, \forall w^\perp \in W^\perp, \forall h \in Hg(V)$$

implies $h\left(W^\perp\right) \subseteq W^\perp \; \forall h \in Hg(V)$, as we wanted to show.

Moreover, since $W_\mathbb{R}$ is stable under the action of C we have

$$\begin{aligned} W^\perp \otimes \mathbb{R} &\cong \{x \in V_\mathbb{R} | \varphi_\mathbb{R}(x, w) = 0 \; \forall w \in W_\mathbb{R}\} \\ &= \{x \in V_\mathbb{R} | \varphi_\mathbb{R}(x, Cw) = 0 \; \forall w \in W_\mathbb{R}\}; \end{aligned}$$

as $(2\pi i)^{-1}\varphi_\mathbb{R}(\cdot, C\cdot)$ is positive defined, this implies that $W_\mathbb{R}$ and $W_\mathbb{R}^\perp$ intersect trivially. Their sum is therefore direct, whence an isomorphism of $\mathbb{Q}$-Hodge structures $V \cong W \oplus W^\perp$.

Finally, the category of polarizable Hodge structures is clearly closed under direct sum and tensor products.

Using polarizations it is not difficult to prove the following simple but very useful results:

Proposition 3.2.8. *Let A be any Abelian variety and H be its Hodge (resp. Mumford-Tate) group. Then H is reductive.*

Proof. In view of Theorem 1.1.17 we only need to show that H admits a faithful semisimple representation. Note that $H_1(A, \mathbb{Q})$ is polarizable, and Remark 3.2.7 implies that the tautological representation

$$Hg(A) \hookrightarrow GL(H_1(A, \mathbb{Q})),$$

which is certainly faithful, is in fact also semisimple: the sub-H-modules are precisely the sub-Hodge structures, and since every sub-Hodge structure admits an (orthogonal) complement the representation is completely reducible. □

Lemma 3.2.9. *Let V be a simple Hodge structure of weight -1 associated to an Abelian variety A, $\varphi : V \otimes V \to \mathbb{Q}(1)$ a polarization of V coming from a polarization ψ of A. Let D be the endomorphism algebra of A, † the Rosati involution associated to ψ, F the center of D and $Z_H = Z(Hg(A))^0$. Let furthermore T_F be the torus $\mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ and U_F be the kernel of $Nm : T_F \to \mathbb{G}_m$, i.e. the subtorus defined by $x\overline{x} = 1$.*

Then $Z_H \subseteq U_F^0$ and $Hg(A) \subseteq Sp(V, \varphi)$.

Proof. We know from Proposition 3.1.9 that

$$(*) \quad D = (\mathrm{End}_{\mathbb{Q}}(V))^{Hg(V)},$$

so $Hg(V)$ commutes with D. At the level of $\mathbb{Q}$-points, $Z_H < Hg(V)$ is then a subset of $\mathrm{End}_{\mathbb{Q}}(V)$ that commutes with $Hg(V)$, so $Z_H \subseteq D$, and moreover $Z_H \subseteq D^*$, since $Z_H < Hg(V) < GL(V)$ contains only invertible elements. Again from $(*)$ we find that D commutes with $Hg(V)$, so in particular it commutes with $Z_H < Hg(V)$, hence Z_H is contained in the center of D. It follows that $Z_H \subseteq F$ and $Z_H \subseteq \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ as algebraic groups (by the density of $\mathbb{Q}$-points).

We can now exploit the fact that φ is a morphism of Hodge structures, which means

$$\varphi(hv \otimes hw) = h\varphi(v \otimes w) = \varphi(v \otimes w) \quad \forall v, w \in V, \forall h \in Hg(V),$$

since $Hg(V)$ acts through $Nm \equiv 1$ on $\mathbb{Q}(1)$.

It follows that Hg preserves the skew-symmetric form associated to φ, so $Hg(V) \subseteq Sp(V, \varphi)$. The Rosati involution is the adjunction with respect to φ, so

$$\varphi(dv \otimes w) = \varphi(v \otimes d^\dagger w) \quad \forall v, w \in V, \forall d \in D;$$

as $Hg(V) < D^*$, the above formula holds in particular for $d \in Hg(V)$, whence

$$\varphi(v \otimes w) = \varphi(hv \otimes hw) = \varphi(v \otimes h^\dagger hw) \quad \forall v, w \in V, \forall h \in Hg(V),$$

and since the bilinear form associated to φ is non-degenerate we find $h^\dagger h = 1$. But on $Hg(V) \subset F$ the Rosati involution coincides with complex conjugation, so $\bar{h}h = 1$ for every h in $Hg(V)$ and we finally find the inclusion $H \subseteq U_F^0$. □

Corollary 3.2.10. *Let A be an Abelian variety without simple factors of type IV. Then $Hg(A)$ is semisimple.*

Proof. We already know (Proposition 3.2.8) that $Hg(A)$ is reductive, whence a decomposition of $Hg(A) \cong Hg(A)' \cdot Z(Hg(A))$ with $Hg(A)'$ semisimple. It is then enough to show that $Z(Hg(A))$ is finite, and it is enough to do so for a simple variety. In this case, $Z(Hg(A))$ is a subgroup of U_F^0 by the above Lemma, so it suffices to show that U_F^0 is finite. Let Σ be the set of embeddings $F \hookrightarrow \mathbb{C}$. Then

$$X^*(T_F) = \bigoplus_{\gamma \in \Sigma} \mathbb{Z}\gamma$$

with its natural action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$, while $X^*(U_F)$ is the quotient of $X^*(T_F)$ by those homomorhpisms that are trivial on the elements of norm 1, i.e. those of the form $\sigma + \overline{\sigma}, \sigma \in \Sigma$.

By the Albert classification, though, F is a totally real field (since no factor is of type IV), so $\overline{\sigma} = \sigma$ and the character group of U_F^0 is trivial, since the character group of U_F is finite, being the quotient of $\bigoplus_{\gamma \in \Sigma} \mathbb{Z}\gamma$ by $\bigoplus_{\sigma \in \Sigma} \mathbb{Z}(2\sigma)$. □

3.3 Product decomposition

We record for later use a general and very useful result on Lie algebras, essentially due to Ribet ([Rib76]):

Lemma 3.3.1. *Let* $\mathbf{C}$ *be an algebraically closed field of characteristic zero and* $V_1, \ldots, V_n$ *be finite-dimensional* $\mathbf{C}$*-vector spaces. Let* $\mathfrak{gl}(V_i)$ *be the Lie algebra of endomorphisms of* V_i *and let* $\mathfrak{g}$ *be a Lie subalgebra of* $\mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_n)$*. For each* $i = 1, \cdots, n$ *let* $\pi_i : \prod_{j=1}^n \mathfrak{gl}(V_j) \to \mathfrak{gl}(V_i)$ *be the* i*-th canonical projection and let* $\mathfrak{g}_i = \pi_i(\mathfrak{g})$*.*

Suppose that each $\mathfrak{g}_i$ *is a simple (nonzero, but possibly Abelian of dimension one) Lie algebra and that one of the following conditions holds:*

(a) For every pair of indices i, j *the projection* $\pi_i \times \pi_j : \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ *is onto.*

(b) For every simple Lie algebra $\mathfrak{l}$ *let*

$$I(\mathfrak{l}) = \{i \in \{1, \cdots, n\} \,|\, \mathfrak{l} \cong \mathfrak{g}_i\}.$$

For every $\mathfrak{l}$ *such that* $|I(\mathfrak{l})| > 1$ *the following conditions are met:*

1. *every automorphism of* $\mathfrak{l}$ *is inner;*
2. *choose isomorphisms* $\varphi_k : \mathfrak{l} \to \mathfrak{g}_k$ *(for* $k \in I(\mathfrak{l})$*). Then the representations of* $\mathfrak{l}$ *induced by the composition of* φ_k *with the tautological representations of* $\mathfrak{g}_k$ *on* V_k*, for* k *varying in* $I(\mathfrak{l})$*, are all isomorphic;*
3. *the equality*

$$\operatorname{End}_{\mathfrak{g}}\left(\bigoplus_{i \in I(\mathfrak{l})} V_i\right) \cong \prod_{i \in I(\mathfrak{l})} \operatorname{End}_{\mathfrak{g}_i} V_i.$$

   holds.

Then $\mathfrak{g} = \prod_{j=1}^{n} \mathfrak{g}_j$*.*

Proof. Let us prove the Lemma under the assumptions of (a). We proceed by induction on n, the case $n = 1$ being trivial.

For $n = 2$ the result follows from simple linear algebra: $\mathfrak{g} \subset \mathfrak{g}_1 \times \mathfrak{g}_2$ forces $\dim(\mathfrak{g}) \leq \dim(\mathfrak{g}_1) + \dim(\mathfrak{g}_2)$, and on the other hand we have a surjective map $\mathfrak{g} \twoheadrightarrow \mathfrak{g}_1 \times \mathfrak{g}_2$, so $\dim(\mathfrak{g}) \geq \dim(\mathfrak{g}_1) + \dim(\mathfrak{g}_2)$, and equality (of dimension, hence equality as vector spaces) must hold.

Let now $n \geq 3$ and $\tilde{I} = \ker(\pi_n : \mathfrak{g} \twoheadrightarrow \mathfrak{g}_n)$. Write $\tilde{I} = I \oplus 0$ for a certain subspace I of $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$.

I is then an ideal of $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$: to see this, let N be its normalizer in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$. We want to show that N fulfills the hypotheses of the Lemma in the case $n-1$, so N equals $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ and I is an ideal of $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$.

- The projections $\pi_i : N \to \mathfrak{g}_i, i = 1, \cdots, n-1$ are surjective: indeed, N contains I, and since the combined projection $\pi_i \times \pi_n : \mathfrak{g} \to \mathfrak{g}_i \oplus \mathfrak{g}_n$ is surjective, for each $g_i \in \mathfrak{g}_i$ we can choose an inverse image a of $(g_i, 0)$ through this double projection. Then clearly $a \in I \subseteq N$ satisfies $\pi_i(a) = g_i$. Note in particular that for every $i = 1, \cdots, n-1$ the projection $\pi_i : I \to \mathfrak{g}_i$ is surjective.

- Let $(g_i, g_j) \in \mathfrak{g}_i \oplus \mathfrak{g}_j$. We want to show that there exists a certain $g \in N$ that projects to (g_i, g_j).

  By hypothesis there is a certain $a \in \mathfrak{g}$ such that $(\pi_i \times \pi_j)(a) = (g_i, g_j)$. Write $a = (g_1, \ldots, g_{n-1}, g_n)$. For any $i = (i_1, \ldots, i_{n-1}) \in I$ we have that $\tilde{i} = (i_1, \ldots, i_{n-1}, 0)$ belongs to $\tilde{I}$, which clearly is an ideal of $\mathfrak{g}$. It follows that $\tilde{I}$ contains

  $$\left[a, \tilde{i}\right] = ([g_1, i_1], \ldots, [g_{n-1}, i_{n-1}], [g_n, 0]) = ([g_1, i_1], \ldots, [g_{n-1}, i_{n-1}], 0),$$

  so I contains $([g_1, i_1], \ldots, [g_{n-1}, i_{n-1}])$. Since this holds for every $i \in I$ we see that $(g_1, \ldots, g_{n-1})$ belongs to N, whence $\pi_i \times \pi_j : N \to \mathfrak{g}_i \oplus \mathfrak{g}_j$ is surjective.

The Lie algebra $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ is clearly semisimple (as it is a direct sum of simple pieces), so its ideal I is in fact a semisimple algebra itself, and more precisely it is of the form

$$\bigoplus_{i \in J} \mathfrak{g}_i$$

for a certain $J \subseteq \{1, \ldots, n-1\}$. But since every projection $\pi_i : I \to \mathfrak{g}_i$ is surjective (as we have already proved) we clearly need to have $J = \{1, \ldots, n-1\}$, whence

$$\dim(\mathfrak{g}) = \dim(I) + \dim(\mathfrak{g}_n) = \sum_{i=1}^{n} \dim(\mathfrak{g}_i),$$

which in turn forces $\mathfrak{g} \cong \bigoplus_{i=1}^{n} \mathfrak{g}_i$.

To prove the lemma with the hypotheses of (b) it suffices to show that (b) implies (a). Let us fix a pair (i, j) and consider the projection $\pi_i \times \pi_j : \mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$. Let $\mathfrak{h}$ be the image of this map: it is a semisimple subalgebra of $\mathfrak{g}_i \times \mathfrak{g}_j$ that projects surjectively on both factors.

As the kernel of the projection $\mathfrak{h} \to \mathfrak{g}_i$ is either trivial or equals $(0) \times \mathfrak{g}_j$ (being isomorphic to an ideal of $\mathfrak{g}_j$), we see that $\mathfrak{h}$ is either $\mathfrak{g}_i \times \mathfrak{g}_j$ or the graph of an isomorphism $\varphi : \mathfrak{g}_i \to \mathfrak{g}_j$. It follows that if $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are not isomorphic, then $\mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is automatically surjective.

Suppose, on the contrary, that $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are isomorphic. We are going to show that $\mathfrak{h}$ still cannot be the graph of an isomorphism $\mathfrak{g}_i \to \mathfrak{g}_j$, since this would contradict the third hypothesis in (b). Suppose by contradiction that this is the case. We want to construct a non-zero morphism of $\mathfrak{g}$-representations

$$\tilde{\chi}_{ij} : V_i \to V_j.$$

Let $\mathfrak{l}$ be an abstract simple Lie algebra to which $\mathfrak{g}_i, \mathfrak{g}_j$ are both isomorphic. The hypotheses imply the existence of the following applications:

- the Lie algebra isomorphism $\varphi : \mathfrak{g}_i \to \mathfrak{g}_j$ whose graph is given by $\mathfrak{h}$;
- Lie algebra isomorphisms $\varphi_i : \mathfrak{l} \to \mathfrak{g}_i$, $\varphi_j : \mathfrak{l} \to \mathfrak{g}_j$;
- an isomorphism of $\mathfrak{l}$-representations $\chi_{ij} : V_i \to V_j$, i.e. an isomorphism of vector spaces such that

$$\chi_{ij}\left(\varphi_i(l) \cdot v_i\right) = \varphi_j(l) \cdot \chi_{ij}\left(v_i\right) \quad \forall v_i \in V_i, \forall l \in \mathfrak{l} \qquad (*).$$

Our assumptions also imply that every $g = (g_1, \ldots, g_n) \in \mathfrak{g}$ satisfies $g_j = \varphi(g_i)$, so the $\tilde{\chi}_{ij}$ we are looking for is a morphism of $\mathfrak{g}$-representations if and only if

$$\tilde{\chi}_{ij}\left(g_i \cdot v_i\right) = \varphi(g_i) \cdot \tilde{\chi}_{ij}\left(v_i\right) \quad \forall v_i \in V_i, \forall g_i \in \mathfrak{g}_i.$$

Choosing $l = \varphi_j^{-1} \circ \varphi(g_i)$ in $(*)$ gives

$$\chi_{ij}\left(\left(\varphi_i \circ \varphi_j^{-1} \circ \varphi\right)(g_i) \cdot v_i\right) = \varphi(g_i) \cdot \chi_{ij}\left(v_i\right).$$

By hypothesis we know that the automorphism $\varphi_i \circ \varphi_j^{-1} \circ \varphi$ of $\mathfrak{g}_i$ is inner, so there exists a certain $a \in GL(V_i)$ such that $\varphi_i \circ \varphi_j^{-1} \circ \varphi(x) = axa^{-1}$. Replacing this expression in the above we get

$$\chi_{ij}\left(a g_i a^{-1} \cdot v_i\right) = \varphi(g_i) \cdot \chi_{ij}\left(v_i\right),$$

hence (choosing $v_i = av$)

$$\left(\chi_{ij} \circ a\right)\left(g_i \cdot v\right) = \varphi(g_i) \cdot \left(\chi_{ij} \circ a\right)(v) \quad \forall v \in V_i, \forall g_i \in \mathfrak{g}_{\mathfrak{i}},$$

so we can take $\chi_{ij} \circ a$ as our $\tilde{\chi}_{ij}$. Now the existence of this $\tilde{\chi}_{ij}$ contradicts the third hypothesis of point (b), since out of it we can fabricate

$$\begin{array}{cccc} \Psi: & \bigoplus_{k\in I(\mathfrak{l})} V_k & \to & \bigoplus_{k\in I(\mathfrak{l})} V_k \\ & (v_{i_1},\cdots,\underbrace{v_i}_{\text{factor } V_i},\cdots,v_{i_{|I(\mathfrak{l})|}}) & \mapsto & (0,\cdots,\underbrace{\tilde{\chi}_{ij}(v_i)}_{\text{factor } V_j},\cdots,0) \end{array}$$

which by construction belongs to $\mathrm{End}_{\mathfrak{g}}\left(\bigoplus_{k\in I(\mathfrak{l})} V_k\right)$, but does not send every factor to itself, so it does not belong to $\prod_{k\in I(\mathfrak{l})} \mathrm{End}_{\mathfrak{g}_k}(V_k)$. This contradiction shows that $\mathfrak{g} \to \mathfrak{g}_i \times \mathfrak{g}_j$ is onto, hence (a) applies and yields the desired conclusion. □

Lemma 3.3.2. *Let A be an Abelian variety isogenous to a product $X_1 \cdots X_n$. Then $Hg(A) \subset Hg(X_1) \times \cdots \times Hg(X_n)$, and its projection on each factor is surjective. Same statement for the Mumford-Tate group.*

Proof. The statements for the Mumford-Tate group and the Hodge group are clearly equivalent, since we already know from Proposition 3.1.6 that $MT(X_i) = \mathbb{G}_m \cdot Hg(X_i)$.

For Mumford-Tate groups everything follows immediately from the definitions: indeed, let $V_i = H_1(X_i, \mathbb{Q})$, $V = H_1(A, \mathbb{Q})$ and

$$\rho_i : \mathbb{S} \to GL(V_{i,\mathbb{R}}), \quad \rho : \mathbb{S} \to GL(V_{\mathbb{R}})$$

be the morphisms defining the Hodge structures on V_i and V respectively. Then $\rho = \bigoplus_{i=1}^n \rho_i$, and each ρ_i factors through its corresponding $MT(X_i)_{\mathbb{R}}$, so ρ factors through $MT(X_1)_{\mathbb{R}} \times \cdots \times MT(X_n)_{\mathbb{R}}$. By minimality of $MT(A)$ we then have $MT(A) \subseteq MT(X_1) \times \cdots \times MT(X_n)$.

Conversely, let M_i be the image of the projection $\pi_i : MT(A) \to GL(V_i)$. As $\bigoplus_{i=1}^n \rho_i$ factors through $MT(A)_{\mathbb{R}}$ and π_i is defined over $\mathbb{Q}$, $(M_i)_{\mathbb{R}}$ factors

$$\pi_i \left(\bigoplus_{k=1}^n \rho_k \right) = \rho_i,$$

which in turn implies (by minimality of $MT(X_i)$) $M_i \supseteq MT(X_i)$, as claimed. □

As an immediate consequence we get a generalization of Proposition 3.1.10, where we drop the assumption that A is simple:

Corollary 3.3.3. *Let A be any Abelian variety over $\mathbb{C}$ with Mumford-Tate group M. Then A has complex multiplication if and only if M is a torus.*

Proof. Let $A \cong A_1 \cdot \ldots \cdot A_n$ be the decomposition of A as product of (possibly repeated) simple factors: then A is of CM type if and only if each factor is.

If $Hg(A)$ is a torus, $Hg(A_i)$ - being a connected quotient of a $Hg(A)$ - is itself a torus, so A_i is of CM type because of Prop. 3.1.10.

If, conversely, every factor A_i admits complex multiplication, then each $Hg(A_i)$ is a torus, so $Hg(A)$ is a subgroup of $Hg(A_1) \times \cdots \times Hg(A_n)$. It follows that it is commutative, connected and reductive, hence a torus. □

Combining Ribet's lemma with the description of the Hodge group of a product we also get the following

Corollary 3.3.4. *Let A_1, A_2 be two Abelian varieties. Suppose that the Lie algebras of $Hg(A_1)$ and $Hg(A_2)$ are semisimple, and all the simple factors appearing in their decomposition as products of simple algebras are pairwise non-isomorphic. Then $Hg(A_1 \times A_2) \cong Hg(A_1) \times Hg(A_2)$.*

Proof. Let H, H_1, H_2 be the Hodge groups of $A := A_1 \times A_2, A_1, A_2$ respectively, and let $\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2$ be the corresponding Lie algebras.

By Lemma 3.3.2 we know that $H \subseteq H_1 \times H_2$ and that the projections $\pi_i : H \to H_i$ of H on each factor are surjective. It follows that for each simple factor $\mathfrak{l}$ of $\mathfrak{h}_1, \mathfrak{h}_2$ the induced maps $\pi_{\mathfrak{l}} : \mathfrak{h} \to \mathfrak{h}_i \to \mathfrak{l}$ are surjective, too, and since the simple factors are pairwise not isomorphic part (b) of Lemma 3.3.1 applies to give $\mathfrak{h} \cong \mathfrak{h}_1 \times \mathfrak{h}_2$, whence (by connectedness) $H = H_1 \times H_2$. □

At the opposite end of the spectrum, we see that the exponent of a simple factor 'does not show up' in the Hodge group:

Lemma 3.3.5. *Suppose the Abelian variety A is isogenous to B^n, where B is simple. Then we can identify $H_1(A, \mathbb{Q}) \cong H_1(B, \mathbb{Q})^{\oplus n}$, and we have $Hg(A) \cong Hg(B)$, where the second group acts diagonally on $H_1(B, \mathbb{Q})^{\oplus n}$.*

More generally, let $A \cong A_1^{n_1} \times \cdots \times A_k^{n_k}$ be the decomposition of A as product of powers of pairwise non-isomorphic simple varieties. Then $Hg(A) \cong Hg\left(A_1 \times \cdots \times A_k\right)$.

Proof. Let $W = H_1(B, \mathbb{Q})$ and $V = H_1(A, \mathbb{Q}) \cong W^{\oplus n}$; let furthermore $\rho : \mathbb{S} \to GL\left(V_{\mathbb{R}}\right)$, $\rho_1 : \mathbb{S} \to GL\left(W_{\mathbb{R}}\right)$ be the morphisms defining the Hodge structures. Clearly $\sigma = \underbrace{\rho \oplus \cdots \oplus \rho}_{n \text{ times}}$, so $Hg(A)$ is contained in the diagonal of $Hg(B)^n$, since ρ factors through the $\mathbb{R}$-points of this last group.

On the other hand, Lemma 3.3.2 implies that $Hg\left(B^n\right) \subset Hg(B)^n$ projects surjectively on each factor $Hg(B)$, so $Hg\left(B^n\right)$ equals the diagonal of $Hg(B)^n$, as claimed.

In the general case, let $V_i = H_1(A_i, \mathbb{Q})$ and H be the image of $Hg(A_1 \times \cdots \times A_k) \subseteq Hg(A_1) \times \cdots \times Hg(A_k)$ in $GL(V_1^{\oplus n_1} \oplus \cdots \oplus V_k^{n_k})$, the immersion being given by the diagonal action on each factor.

The same argument as above shows that H contains the Hodge group of A, since the defining morphism of $H_1(A, \mathbb{Q})$ factors through the real points of H.

On the other hand, for $i = 1, \cdots, k$ let B_i be a distinguished copy of the simple factor A_i; then the projection of $Hg(A) \subseteq \operatorname{diag}(Hg(A_1)^{n_1}) \times \cdots \times \operatorname{diag}(Hg(A_k)^{n_k}) \to Hg(B_1) \times \cdots \times Hg(B_k)$ factors the defining morphism of $V_1 \oplus \cdots \oplus V_k$, so it contains the Hodge group of $Hg(B_1 \times \cdots \times B_k)$. It follows that the Hodge group of A cannot be smaller than H and we have the desired equality. □

3.4 On bilinear forms and Hodge groups

It turns out to be way more convenient to work with bilinear forms instead of using divisors directly.

We start by establishing a few general properties of bilinear forms before turning to the connection they bear to our study of Abelian varieties.

The general picture we will be interested in is as follows: let F_2/F_1 be a separable field extension. Suppose we are given a finite-dimensional vector space V over F_2. Then V inherits a structure of F_1-vector space, and we would like to investigate the relations between F_1- and F_2-linear forms on V. The following constructions are essentially taken from [Del79].

A first general result is the following:

Proposition 3.4.1. *For every F_2-vector space V of finite dimension we have a natural identification*

$$\begin{array}{ccc} \operatorname{Hom}_{F_2}(V, F_2) & \longrightarrow & \operatorname{Hom}_{F_1}(V, F_1) \\ \chi & \mapsto & \operatorname{tr}_{F_2/F_1} \circ \chi \end{array}$$

Proof. As F_2/F_1 is separable, the pairing

$$(x, y) \mapsto \operatorname{tr}_{F_2/F_1}(xy)$$

is nondegenerate, so the above map is injective, and it is surjective because the two spaces have the same dimension over F_1. □

With this at hand we can easily determine which F_1-bilinear forms are actually induced by taking traces from F_2 to F_1:

Proposition 3.4.2. *Let V, W be vector spaces over F_2 and $\varphi : V \times W \to F_1$ an F_1-bilinear form. Then there exists a F_2-bilinear form $\psi : V \times W \to F_2$ such that $\varphi = \operatorname{tr}_{F_2/F_1} \psi$ if and only if*

$$\varphi(f_2 v, w) = \varphi(v, f_2 w) \quad \forall f_2 \in F_2, v, w \in V.$$

When ψ exists, it is unique. Finally, if $W = V$ and φ is symmetric (resp. skew-symmetric), so is ψ.

Proof. The condition is clearly necessary.

To see that it is also sufficient, note that it implies that we can think of φ as an F_1-linear form

$$\varphi : V \otimes_{F_2} W \to F_1,$$

so - by the above Proposition - there is a unique F_2-bilinear $\psi : V \otimes_{F_2} W \to F_2$ such that $\varphi = \operatorname{tr}_{F_2/F_1} \circ \psi$.

For the final statement, simply notice that if S denotes the symmetrization (resp. anti-symmetrization) operator, then

$$\varphi = S\varphi = S\left(\operatorname{tr}_{F_2/F_1} \psi\right) = \operatorname{tr}_{F_2/F_1}(S\psi),$$

where S commutes with tr by linearity. The uniqueness property then implies $S\psi = \psi$, so ψ is a fixed point for S whenever φ is. □

In order to show the relevance of the above for our study of Abelian varieties we now introduce a rather natural description of divisor classes in terms of alternating bilinear forms. Let $A = V/\Lambda$ be a complex Abelian variety. Given a divisor $D \in Pic(A)$, we can take its Chern class, getting an element of $H^2(A, \mathbb{Z})$, which we then identify to $\operatorname{Hom}\left(\bigwedge^2 \Lambda, \mathbb{Z}\right)$. Finally, extending scalars to $\mathbb{R}$ yields a bilinear, alternating form $\delta : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$ associated to D. The following is a characterization of the bilinear forms that arise in this way:

Proposition 3.4.3. *Let A be an Abelian variety over $\mathbb{C}$ equipped with a polarization $\varphi : A \to A^\vee$. Let $e \mapsto e^\dagger$ be the Rosati involution associated with the given polarization and $NS(A)$ be the Néron-Severi group of A.*

Then the map

$$\begin{array}{rcccccc} \chi : & NS(A) \otimes_{\mathbb{Z}} \mathbb{Q} & \to & Hom^0(A, A^\vee) & \to & End^0(A) \\ & [\mathcal{M}] & \mapsto & \varphi_{\mathcal{M}} & \mapsto & \varphi_{\mathcal{L}}^{-1} \circ \varphi_{\mathcal{M}} \end{array}$$

identifies $NS(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ with the space

$$\left\{ e \in \operatorname{End}^0(A) | e = e^\dagger \right\}$$

of $\dagger$-symmetric endomorphisms of A.

Proof. Write $A = V/\Lambda$, $A^\vee = \overline{V}^*/\Lambda^\vee$ and let

$$\Phi_{\mathcal{L}} : V \to \overline{V}^*$$

be the analytic representation of $\varphi_{\mathcal{L}}$. We have an Hermitian form $H : V \times V \to \mathbb{C}$, given by

$$H\left(v_1, v_2\right) = \left(\Phi_{\mathcal{L}}(v_1)\right)(v_2).$$

Recall that the Rosati involution is explicitly given by

$$\psi \mapsto \varphi_{\mathcal{L}}^{-1} \circ \psi^\vee \circ \varphi_{\mathcal{L}},$$

and that it gives the adjoint map for H: identifying an element $e \in End^0(A)$ with its analytic representation we have

$$\begin{aligned} H\left(e^\dagger v_1, v_2\right) &= \left(\varphi_{\mathcal{L}} \circ \varphi_{\mathcal{L}}^{-1} \circ e^\vee \circ \varphi_{\mathcal{L}}(v_1)\right)(v_2) = \left(e^\vee \circ \varphi_{\mathcal{L}}(v_1)\right)(v_2) \\ &= \left(\varphi_{\mathcal{L}}(v_1)\right)(ev_2) = H(v_1, ev_2). \end{aligned}$$

Now a rational endomorphism e is $\dagger$-symmetric if and only if

$$H(ev_1, v_2) = H(v_1, e^\dagger v_2) = H(v_1, ev_2) = \overline{H(ev_2, v_1)},$$

that is to say, exactly when the bilinear form

$$(v_1, v_2) \mapsto H(ev_1, v_2)$$

is Hermitian. Explicitly, the above form is given by

$$(v_1, v_2) \mapsto \left(\Phi_{\mathcal{L}} \circ e(v_1)\right)(e_2),$$

and it is a well-known result (see, for example, Theorem 2.5.5 of [BL04]) that this form is Hermitian exactly when $\Phi_{\mathcal{L}} \circ e$ is the analytic representation of $\varphi_{\mathcal{M}}$ for a certain line bundle $\mathcal{M}$. Putting everything together, we see that e is $\dagger$-symmetric if and only if $e = \Phi_{\mathcal{L}}^{-1} \circ \Phi_{\mathcal{M}}$, i.e. exactly when it belongs to the image of χ. □

In view of our identification of Chern classes with bilinear forms, we can restate the above Proposition simply as

Theorem 3.4.4. *Let A be a polarized Abelian variety over $\mathbb{C}$ and let φ be the non-degenerate, bilinear form associated with the polarization. Then the cohomology class of a divisor can be identified with a bilinear form $\varphi(e\cdot, \cdot)$ for a certain $\dagger$-symmetric rational endomorphism e of A.*

Finally, we turn our attention to Hermitian forms; we work directly in the setting of Abelian varieties (see [Del79], Lemmas 4.6 and 4.7).

Let A be a polarized Abelian variety over $\mathbb{C}$ (resp. a number field K), $V := H_1(A, \mathbb{Q})$, F a CM-field and $\nu : F \hookrightarrow End^0(A)$ an injective ring homomorphism. In particular, this means $\nu(1) = 1_A$. Suppose that the polarization on A is chosen in such a way that the associated Rosati involution induces 'complex conjugation' on F (i.e. the unique nontrivial automorphism $\alpha \mapsto \alpha'$ of F over its totally real maximal subfield). This is always possible, as already remarked. With the above notation we have

Proposition 3.4.5. *Fix any nonzero $\alpha \in F$ with $\alpha' = -\alpha$ and denote by φ the $\mathbb{Q}$-bilinear form on V induced by the given polarization. Then there exists a unique F-Hermitian form*

$$\psi : V \times V \to F$$

such that $\varphi(v, w) = tr_{F/\mathbb{Q}}(\alpha\psi(v, w))$.

Proof. We apply Proposition 3.4.2 with $V = W = H_1(A, \mathbb{Q})$, $F_1 = \mathbb{Q}$ and $F_2 = F$, where the action of F is the natural one on V and is through complex conjugation on W. This gives a first form ψ_1 such that $\varphi = tr_{F/\mathbb{Q}}(\psi_1)$; let $\psi = \alpha^{-1}\psi_1$, so that we trivially have $\varphi = tr_{F/\mathbb{Q}}(\alpha\psi)$.

We only need check that ψ thus defined is in fact Hermitian. It follows from the definitions that ψ_1 (and hence ψ) is sesquilinear (note that the action of F on the second factor is through complex conjugation), so it suffices to show that $\psi(v, w) = \psi(w, v)'$. We have $\operatorname{tr}_{F/\mathbb{Q}}(\alpha\psi(y, x)) = \varphi(y, x) = -\varphi(x, y) = -\operatorname{tr}_{F/\mathbb{Q}}(\alpha\psi(x, y))$, so

$$\operatorname{tr}_{F/\mathbb{Q}}(\alpha\psi(x, y)) = \operatorname{tr}_{F/\mathbb{Q}}(-\alpha\psi(y, x)) = \operatorname{tr}_{F/\mathbb{Q}}\left(\alpha'\psi(y, x)\right).$$

On the other hand, by F-linearity, on replacing x with fx for an $f \in F$, we have

$$\begin{aligned}\operatorname{tr}_{F/\mathbb{Q}}(\alpha f\psi(x, y)) &= \operatorname{tr}_{F/\mathbb{Q}}(\alpha\psi(fx, y)) = \operatorname{tr}_{F/\mathbb{Q}}\left(\alpha'\psi(y, fx)\right) \\ &= \operatorname{tr}_{F/\mathbb{Q}}\left(\alpha' f'\psi(y, x)\right) = \operatorname{tr}_{F/\mathbb{Q}}\left((\alpha f)'\psi(y, x)\right);\end{aligned}$$

finally, since $\operatorname{tr}_{F/\mathbb{Q}}(\beta') = \operatorname{tr}_{F/\mathbb{Q}}(\beta)\ \forall \beta \in F$, we get

$$tr_{F/\mathbb{Q}}(\alpha f\psi(x, y)) = \operatorname{tr}_{F/\mathbb{Q}}\left((\alpha f\psi(x, y))'\right),$$

hence - on comparing the different expressions for $\operatorname{tr}_{F/\mathbb{Q}}(\alpha f\psi(x, y))$ - we find

$$\operatorname{tr}_{F/\mathbb{Q}}\left((\alpha f\psi(x, y))'\right) = \operatorname{tr}_{F/\mathbb{Q}}\left((\alpha f)'\psi(y, x)\right).$$

As f ranges through F, $(f\alpha)'$ does the same, so - as $\operatorname{tr}_{E/\mathbb{Q}}$ is non-degenerate - $\psi(y, x) = (\psi(x, y))'$, as we wanted to show.

Finally, the uniqueness part follows from the analogous statement in Proposition 3.4.2. □

Let's now apply the above to the case at hand: let A be a simple Abelian variety, $V = H_1(A, \mathbb{Q})$ and φ be the bilinear form on V induced by a polarization. Suppose we are given a number field E together with an inclusion of rings $i : E \hookrightarrow \mathrm{End}^0(A) =: D$.

We single out two particular and very important cases:

- *E is totally real.*

  By Proposition 3.4.2 there exists a unique E-bilinear form

  $$\psi : V \times V \to E$$

  that induces φ. A key fact in all that follows is the following:

  Proposition 3.4.6. *The Hodge group $H := Hg(A)$ preserves ψ.*

  Proof. Take any $h \in H$ and consider the bilinear form

  $$\psi_h(v, w) := \psi(hv, hw).$$

  On one hand, $tr_{E/\mathbb{Q}}(\psi_h(v, w)) = \varphi(hv, hw) = \varphi(v, w)$, since the Hodge group preserves the forms induced by polarizations.

  On the other hand, the Hodge group commutes with E by Proposition 3.1.9, so

  $$\psi_h(e_1 v, e_2 w) = \psi(he_1 v, he_2 w) = \psi(e_1 hv, e_2 hw)$$
  $$= e_1 e_2 \psi(hv, hw) = e_1 e_2 \psi_h(v, w)$$

  for every choice of $e_1, e_2 \in E$, hence ψ_h is E-bilinear. By uniqueness of ψ this implies $\psi_h = \psi$, so the Hodge group preserves ψ, as we wanted to show. □

  It is thus very natural to introduce the following

  Definition 3.4.7. Suppose we have fixed a polarization on A once and for all. It then makes sense to define $Sp\,(V/E)$ to be the ($\mathbb{Q}$-algebraic) group of E-automorphisms of V that preserve ψ. We shall make frequent use of $\mathfrak{sp}\,(V/E)$, the Lie algebra of this group. Explicitly,

  $$\mathfrak{sp}\,(V/E) = \{e \in \mathrm{End}_E(V) | \psi(ev, w) + \psi(v, ew) = 0 \; \forall v, w \in V\}.$$

- *E is a CM field.* Let E_0 be the maximal totally real subfield of E and $e \mapsto e'$ be the Rosati involution associated to φ. We can (and will) always choose φ in such a way that the Rosati involution induces complex conjugation on E. For any choice of a non-zero $\alpha \in E$ with

$\alpha' = -\alpha$, Proposition 3.4.5 yields the existence of a unique E-Hermitian form ψ such that $\varphi(v,w) = \operatorname{tr}_{E/\mathbb{Q}}(\alpha\psi(v,w))$.

The same proof as in the case of totally real fields again shows that the Hodge group preserves ψ, and so its Lie algebra is contained in $\{e \in \operatorname{End}_E(V) | \psi(ev,w) + \psi(v,ew) = 0 \; \forall v,w \in V\}$, that we shall denote $\mathfrak{u}(V/E)$.

Remark 3.4.8. In the case E is a real field, the same arguments as above also prove that $Sp(V/E) = Res_{E/\mathbb{Q}}(Sp(V,\psi))$. Similarly, noticing that $U(V,\psi)$ is an algebraic group over E_0 and not over E in the CM case (for the same reason why the unitary group is just a *real* Lie group and not a complex one), $U(V/E) = \operatorname{Res}_{E_0/\mathbb{Q}}(U(V,\psi))$.

Finally, a simple variant of the argument for Hermitian forms works even when D is a quaternion algebra (the key property is that the pairing induced by the trace form is nondegenerate), in which case it will be useful to know that φ determines a 'skew-D-Hermitian' form, i.e. a form $\psi : V \times V \to D$, D-linear in the first argument, that induces φ through the trace, and such that $\varphi(w,v) = -\varphi(v,w)'$.

All of the above can also be repeated in an ℓ-adic setting. Let us consider the case where E is a totally real field. Let A be an Abelian variety over a number field K, $T_\ell := T_\ell(A)$ be the Tate module of A, $V_\ell := T_\ell \otimes \mathbb{Q}_\ell$ and suppose that E admits a (ring) embedding in $\operatorname{End}^0(A)$. A polarization on A also induces a bilinear form $\varphi_\ell : V_\ell \times V_\ell \to \mathbb{Q}_\ell$, and V_ℓ is a free module of rank $\frac{2\dim(A)}{[E:\mathbb{Q}]}$ over $E_\ell := E \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. In this case we obtain a unique E_ℓ-bilinear form ψ_ℓ on V_ℓ such that $\varphi_\ell = \operatorname{tr}_{E_\ell/\mathbb{Q}_\ell}(\psi_\ell)$ and a corresponding Lie algebra $\mathfrak{sp}_{E_\ell}(V_\ell,\psi_\ell)$, regarded as a Lie algebra over $\mathbb{Q}_\ell$.

Finally, in case E is a CM field, for any choice of $\alpha \in E$ such that $\alpha' = -\alpha$ we find a unique ψ_ℓ such that $\varphi_\ell = \operatorname{tr}_{E_\ell/\mathbb{Q}_\ell}(\alpha\psi_\ell)$, and the relevant Lie algebra is $\mathfrak{u}_{E_\ell}(V_\ell,\psi_\ell)$ regarded over $\mathbb{Q}_\ell$.

Finally, we introduce here the definition of the Lefschetz group and prove a basic property (taken from [Mur84], Lemma 2.1) we are going to need later:

Definition 3.4.9. Let A be a (not necessarily simple) polarized Abelian variety defined over $\mathbb{C}$, $V := H_1(A,\mathbb{Q})$, φ the bilinear alternating form induced on V by the polarization and $D := \operatorname{End}^0(X)$ the endomorphism algebra.

The **Lefschetz group** $L(A)$ is the connected component of the identity of the centralizer of D inside $\operatorname{Aut}_D(V,\varphi)$, the group of D-automorphisms of V preserving φ.

Remark 3.4.10. Note that since two polarizations differ by the action of an endomorphism of A, the group $L(A)$ does not depend on the choice of the polarization. Moreover, it is clear from the definition and the above considerations on bilinear forms that $Hg(A)$ is always a subgroup of $L(A)$.

Finally, $L(A)$ is, in many particular cases, contained in other groups we have introduced: for example, if A is simple and of Type I, then $L(A)$ is a subgroup of $Sp(V/D)$, and if A is simple and D contains a CM field E, then $L(A)$ is a subgroup of $U(V/E)^0$.

Proposition 3.4.11. *Let $A \cong B_1^{n_1} \cdot \ldots \cdot B_k^{n_k}$ be the decomposition of A as product of powers of pairwise non-isomorphic simple Abelian varieties B_i. Then $L(A) \cong L(B_1) \times \cdots \times L(B_k)$.*

Proof. For $i = 1, \ldots, k$ let $A_i := B_i^{n_i}$ and V_i be $H_1(A_i, \mathbb{Q})$. Choose polarizations $\psi_1, \ldots, \psi_k$ on $V_1, \ldots, V_k$ and note that $\psi := \psi_1 \oplus \cdots \oplus \psi_k$ is a polarization of $H_1(A, \mathbb{Q})$. As

$$\operatorname{End}(A) \otimes \mathbb{Q} \cong \prod_{i=1}^{k} \operatorname{End}^0(A_i),$$

an automorphism of A must preserve each V_i, so that preserving ψ is equivalent to preserving ψ_i for each i, which shows $L(A) \cong \prod_{i=1}^{k} L(A_i)$. Now fix an index i, consider $W = H_1(B_i)$ and fix a polarization $\tilde{\psi}$ on W. We have $V_i = W^{\oplus n_i}$ and we can take ψ_i to be $\underbrace{\tilde{\psi} \oplus \cdots \oplus \tilde{\psi}}_{n_i \text{ times}}$. As $\operatorname{End}(A_i) = \operatorname{Mat}_{n_i}\left(\operatorname{End}^0(B_i)\right)$, an operator commuting with the full endomorphism algebra must act in the same way on each factor W, thus identifying the centralizer of $\operatorname{End}(A_i)$ in $Sp(V, \psi)$ to the centralizer of $\operatorname{End}(B_i)$ in $Sp\left(W, \tilde{\psi}\right)$. □

CHAPTER 4

Two worked out examples

We start from the two simplest classes of Abelian varieties, elliptic curves and Abelian surfaces, and work out which groups can be realized as $MT(A)$ in these cases.

This chapter basically consists of an elaboration of Examples 5.4 and 5.7 and Exercise 5.6 of [Mooa], filling in the missing details in the hope of giving a complete and almost self-contained proof of the classification result in these cases.

4.1 Mumford-Tate groups of Elliptic Curves

Let E be an elliptic curve over $\mathbb{C}$, and represent it as a quotient $E = \mathbb{C}/\Lambda$ of $\mathbb{C}$ by a full-rank lattice.

As we already know (Proposition 3.2.8) that $Hg(E)$ is a reductive group, we start by classifying the reductive subgroups of SL_2:

Lemma 4.1.1. *Let k be an algebraically closed field. The only connected reductive subgroups of $SL_{2,k}$ are then the trivial group, $SL_{2,k}$ itself and the maximal tori.*

Proof. Let G be any such reductive subgroup. We can check what happens in every possible dimension:

Dimension 0 A connected group of dimension zero is clearly trivial.

Dimension 1 Over an algebraically closed field, the only groups of dimension 1 are $\mathbb{G}_a$ and $\mathbb{G}_m$. G cannot be isomorphic to $\mathbb{G}_a$, since the category of representations of this last group is not semisimple: indeed, $\mathbb{G}_a$ admits the two-dimensional representation

$$\begin{array}{ccc} \mathbb{G}_a & \hookrightarrow & SL_2 \\ b & \mapsto & \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}, \end{array}$$

that is not completely reducible. This contradicts Theorem 1.1.17, so G cannot be isomorphic to $\mathbb{G}_a$. It follows that G is a torus, and moreover it is maximal, since the rank of SL_2 is one.

Dimension 2 Consider H, the radical of G. H is a torus by Theorem 1.1.16, hence $\dim(H) \leq \operatorname{rank}(G) \leq \operatorname{rank}(SL_2) = 1$, so G/H has dimension 2 or 1. Let $\mathfrak{j}$ be the Lie algebra of G/H. On one hand, $\mathfrak{j}$ is semisimple, since G/H is semisimple by definition of H.

On the other hand, $\dim(\mathfrak{j}) \leq 2$, and it is easy to check that any Lie algebra of dimension at most 2 is solvable. But the only Lie algebra that is at the same time solvable and semisimple is the trivial one, whence the contradiction $\dim(\mathfrak{j}) = 0$. Therefore SL_2 has no reductive subgroup of dimension 2.

Dimension 3 The scheme SL_2 is irreducible, so every proper subscheme has dimension strictly less than three. It follows that $G < SL_2, \dim(G) = \dim(SL_2)$ implies $G = SL_2$, so this is the only possibility for $\dim(G) = 3$.

□

With the previous Lemma at hand we can now deal with the task of determining the possible Mumford-Tate groups for elliptic curves. As it is well-known (for example as a corollary of the Albert classification, but it is in fact a much simpler result), the endomorphism algebra of E is either $\mathbb{Q}$ or an imaginary quadratic field (in which case E admits complex multiplication). The two cases actually correspond to different Mumford-Tate groups, and the complete result is as follows:

Proposition 4.1.2. *Let E be an elliptic curve over $\mathbb{C}$. Then $Hg(E) = L(E)$. More concretely, there are exactly two possibilities:*

- *E does not admit complex multiplication, in which case $Hg(E) = SL_{2,\mathbb{Q}}$ and $MT(E) = GL_{2,\mathbb{Q}}$;*
- *E admits complex multiplication by an imaginary quadratic field F, in which case*
$$Hg(E) = \left\{x \in \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F} | x\overline{x} = 1\right\}$$
and $MT(E)$ is the almost-direct product $Hg(E) \cdot \mathbb{G}_m$ inside $GL_{2,\mathbb{Q}}$, so $MT(E) = \operatorname{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$.

Proof. Thanks to Lemma 4.1.1, combined with the fact that $Hg(E)$ is reductive, we know that the only possibilities for $Hg(E)$ are the trivial group, a rank-one torus or the full group SL_2.

Suppose first that the endomorphism algebra of E is just $\mathbb{Q}$.

$Hg(E)$ is clearly non trivial, for otherwise Proposition 3.1.9 would yield $\operatorname{End}^0(E) = (\operatorname{End}(V))^{Hg(E)} = \operatorname{End}(V) \neq \mathbb{Q}$, contradiction.

$Hg(E)$ cannot be a torus, either, since otherwise Proposition 3.1.10 would imply that E has complex multiplication, which is against the hypothesis. It follows that $Hg(E) = SL_2$, and $MT(E) = \mathbb{G}_m \cdot Hg(E)$ is the full group GL_2.

Finally, the inclusions $SL_2 = Hg(E) \subseteq L(E) \subseteq SL_2$ prove that $Hg(E) = L(E)$.

Suppose, on the contrary, that E has complex multiplication by an imaginary quadratic field F. We know from Proposition 3.1.10 that in this case the Mumford-Tate group of E (equivalently, its Hodge group) is commutative. Lemma 3.2.9 then yields

$$Hg(E) = Z(Hg(E)) \subseteq U_F = \left\{ x \in \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F} | x\overline{x} = 1 \right\},$$

which is a torus of dimension one. Indeed, let $\sigma, \overline{\sigma}$ be the embeddings of F into $\mathbb{C}$. Then the character group of $T_F := \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_{m,F}$ is free Abelian of rank two, on the generators $\sigma, \bar{\sigma}$, and

$$X^*(U_F) = \frac{X^*(T_F)}{\{\text{homomorphisms trivial on those } x \text{ such that } x\overline{x} = 1\}}$$

$$\cong \frac{\mathbb{Z}\sigma \oplus \mathbb{Z}\overline{\sigma}}{\mathbb{Z}\left(\sigma + \overline{\sigma}\right)},$$

so $X^*(U_F)$ is free of rank 1 and $\dim(U_F) = 1$. By rank considerations it follows that $Hg(E)$, being connected, is either trivial or the full torus U_F. The first case is impossible: if V denotes $H_1(E, \mathbb{Q})$, Proposition 3.1.9 yields

$$D = \operatorname{End}(V)^{Hg(E)} = \operatorname{End}(V),$$

which is absurd, since D is commutative and $\operatorname{End}(V)$ is not. We conclude that $Hg(E) = U_F$, hence

$$MT(E) \cong Hg(E) \cdot \mathbb{G}_m \cong U_F \cdot \mathbb{G}_m = T_F,$$

where the last equality holds because T_F clearly contains U_F and $\mathbb{G}_m$, and both $\mathbb{G}_m \cdot U_F$ and T_F are tori of rank 2.

Finally, note that $L(E)$ commutes with the action of U_F, which is a maximal torus in SL_2, so $L(E)$ must coincide with U_F, hence with $Hg(E)$. □

4.2 Mumford-Tate groups of Abelian Surfaces

We now want to classify which groups can arise as $MT(A)$ for A an Abelian surface. Throughout this section let A be a complex Abelian variety of dimension 2, $V = H_1(A, \mathbb{Q})$, M the Mumford-Tate group of A, $h : \mathbb{S} \to V_{\mathbb{R}}$ the morphism defining the Hodge structure on V. Clearly, we can restrict our attention to the Hodge group of V instead of the full Mumford-Tate group.

$Hg(A)$ again depends on the endomorphism algebra of A, and we have a few more different possibilities. The complete classification is as follows (Example 2.7 in [Moob], but here we try to give a little more detail and cover the reducible case):

Theorem 4.2.1. *Let A be an Abelian surface over $\mathbb{C}$ and D its endomorphism algebra. Then $Hg(A)$ and $L(A)$ coincide.*

More concretely, there are two possibilities for non-simple surfaces:

- *A is isogenous to the self-product of an elliptic curve E, in which case $Hg(A) \cong Hg(E)$;*
- *A is isogenous to the product of two non-isogenous elliptic curves E_1 and E_2, in which case $Hg(A) \cong Hg(E_1) \times Hg(E_2)$*

and four possibilities for simple surfaces:

- *$D = \mathbb{Q}$: then $Hg(A) \cong Sp_{4,\mathbb{Q}}$;*
- *$D = F$, a real quadratic field: then $Hg(A) \cong \mathrm{Res}_{F/\mathbb{Q}}(SL_{2,F})$;*
- *D is a totally indefinite quaternion algebra: then*

$$Hg(A) \cong (D^{*,opp})^{der},$$

where we regard D^* as an algebraic group over $\mathbb{Q}$ (see Section 1.1.6);

- *D is a CM field F of degree 4 over $\mathbb{Q}$: then (with the notation of Lemma 3.2.9) $Hg(A) \cong U_F$.*

The rest of this chapter is dedicated to proving the above Theorem.

4.2.1 The reducible case

Suppose first that A is not irreducible, so that A is isogenous to a product of two elliptic curves E_1, E_2.

Lemma 4.2.2. *Suppose E_1, E_2 have complex multiplication and let M_1, M_2 be their Mumford-Tate groups. Then $M_1 \cong M_2$ if and only if E_1 is isogenous to E_2.*

Proof. The 'if' part is trivial.

For $i = 1, 2$ write $E_i = \mathbb{C}/\langle 1, \tau_i \rangle$ where $\Im\tau_i > 0$. The endomorphism algebra F_i of E_i is then the quadratic imaginary field $\mathbb{Q}(\tau_i)$. We know from the discussion on elliptic curves that

$$M_i \cong \mathrm{Res}_{F_i/\mathbb{Q}}(\mathbb{G}_{m,F_i}),$$

so we can recover F_i from M_i as the group of its $\mathbb{Q}$-valued points.

But then $M_1 \cong M_2$ implies $F_1 = F_2 \subset \mathbb{C}$ (both fields are Galois over $\mathbb{Q}$, so their image in $\mathbb{C}$ is well-defined), from which follows the existence of $a, b \in \mathbb{Q}$ such that $\tau_1 = a\tau_2 + b$. Clearing denominators we have $c\tau_1 = d\tau_2 + e$ for certain integers c, d, e, which in turn means that the two lattices defining E_1, E_2 are commensurable, i.e. E_1 and E_2 are isogenous. □

It turns out to be more convenient to work with Hodge groups; to determine $Hg(A)$ we further distinguish two subcases:

- E_1 is isogenous to E_2, so A is isogenous to E^2 for a certain elliptic curve $E = \mathbb{C}/\Lambda$: Lemma 3.3.5 applies and yields $Hg\left(E^2\right) \cong Hg(E)$ with its diagonal action.

- E_1 and E_2 are not isogenous. Then the Hodge groups of E_1 and E_2 can only be isomorphic if neither has complex multiplication (if this were not the case, then clearly both would have complex multiplication, hence by the above Lemma 4.2.2 E_1 and E_2 would be isogenous, contradiction), in which case $Hg(E_1) = Hg(E_2) = SL_2$. We are then left with the following three cases (up to symmetry): $Hg(E_1) \cong Hg(E_2) \cong SL_2$; $Hg(E_1) \cong SL_2$ and $Hg(E_2)$ is a torus; $Hg(E_1)$ and $Hg(E_2)$ are both tori.

  In the first two cases, part (b) of Lemma 3.3.1 applies (using that $\mathfrak{sl}_2$ only has one irreducible representation of dimension 2, up to isomorphism) and we have $Hg(E_1 \times E_2) = Hg(E_1) \times Hg(E_2)$.

  In the third case we want to show that we still have $Hg(E_1 \times E_2) \cong Hg(E_1) \times Hg(E_2)$, and in order to do so we use the equivalence of categories of Theorem 1.1.1. Write $M_i \cong \mathrm{Res}_{F_i/\mathbb{Q}}(\mathbb{G}_{m,F_i})$ and let $\sigma_i, \overline{\sigma_i}$ be the embeddings of F_i in $\mathbb{C}$. We know from the case of elliptic curves that

$$X^*\left(Hg(E_i)\right) \cong \frac{\mathbb{Z}\sigma_i \oplus \mathbb{Z}\overline{\sigma_i}}{\mathbb{Z}\left(\sigma_i + \overline{\sigma_i}\right)},$$

  so $Hg(E_1) \times Hg(E_2)$ corresponds to the subgroup

$$A := \mathbb{Z}\left(\sigma_1 + \overline{\sigma_1}\right) \oplus \mathbb{Z}\left(\sigma_2 + \overline{\sigma_2}\right)$$

of

$$B := X^*(M_1 \times M_2) \cong \bigoplus_{i=1,2} (\mathbb{Z}\sigma_i \oplus \mathbb{Z}\overline{\sigma_i}).$$

We can now classify the subtori of $Hg(E_1) \times Hg(E_2)$ by studying the subgroups of B containing A. Let C be the (free) subgroup corresponding to $Hg(E_1 \times E_2)$.

If $rk(C) = 4$, then $Hg(E_1 \times E_2)$ is trivial, and this is absurd.

If $rk(C) = 2$ then $C = A$ and $Hg(E_1 \times E_2) = Hg(E_1) \times Hg(E_2)$, as desired.

Finally, if $rk(C) = 3$, then C is generated by $\sigma_1 + \overline{\sigma_1}, \sigma_2 + \overline{\sigma_2}$ and a third element ω, that we can write as $a\sigma_1 + b\sigma_2$. As F_1, F_2 are linearly disjoint over $\mathbb{Q}$ (they are Galois extensions with trivial intersection), there is an element $\chi \in \operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ such that $\chi \cdot \sigma_1 = \overline{\sigma_1}$ and $\chi \cdot \sigma_2 = \sigma_2$, so

$$C \ni \chi \cdot \omega + \omega - a(\sigma_1 + \overline{\sigma_1}) = 2b\sigma_2.$$

Similarly, C contains $2a\sigma_1$, so exactly one between a and b is zero (for otherwise we would have $rk(C) = 4$ or $rk(C) = 2$). We can suppose without loss of generality that $a \neq 0$. On the other hand, the surjective projection $Hg(E_1 \times E_2) \to Hg(E_1)$ induces an injection

$$\begin{array}{ccc} X^*(Hg(E_1)) & \hookrightarrow & X^*\left(Hg(E_1 \times E_2)\right) \cong A/C \\ \sigma & \mapsto & [\sigma], \end{array}$$

and this clearly contradicts $2a[\sigma] = 0$. Therefore the rank of C cannot be 3 and the claim follows.

Finally, note that Proposition 3.4.11 applies to both the above cases and yields the equality $Hg(A) = L(A)$.

4.2.2 The irreducible case

We proceed by treating the different possibilities for $D = \operatorname{End}^0(A)$; that the cases listed in the Theorem are in fact the only possibilities follows by the Albert classification (Theorem 1.3.2 and the following Remark), and we now analyze each of them separately. Note that, in each case, the equality $Hg(A) = L(A)$ will follow easily, once the first group has been described. For the remainder of this chapter, let H denote $Hg(A)$.

4.2.2.1 The case $D = \mathbb{Q}$

We know that H is semisimple and contained in $Sp(V, \varphi) \cong Sp_{4,\mathbb{Q}}$ (by Lemma 3.2.9). Extend scalars to $\mathbb{C}$ and consider the Lie algebra $\mathfrak{h}$ of $Hg(A)_{\mathbb{C}}$. $V_{\mathbb{C}}$ is then an irreducible representation of $\mathfrak{h}$, since

$$\operatorname{End}(V_{\mathbb{C}})^{\mathfrak{h}} \cong \operatorname{End}(V)^{\operatorname{Lie}(Hg(A))} \otimes \mathbb{C} \cong \mathbb{Q} \otimes \mathbb{C}.$$

The rank of $\mathfrak{sp}_{4,\mathbb{C}}$ is 2, hence the rank of $\mathfrak{h} \subseteq \mathfrak{sp}_{4,\mathbb{C}}$ is at most 2. By semisimplicity of $\mathfrak{h}$ and the classification of simple Lie algebras we see that the only possibilities are $\mathfrak{h} \cong \mathfrak{sl}_{2,\mathbb{C}}, \mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{sl}_{2,\mathbb{C}}, \mathfrak{sp}_{4,\mathbb{C}}$ or $\mathfrak{g}_2$.

We want to show that in fact $\mathfrak{h} \cong \mathfrak{sp}_{4,\mathbb{C}}$.

- $\mathfrak{h}$ cannot be isomorphic to $\mathfrak{sl}_{2,\mathbb{C}}$: $V_{\mathbb{C}}$ would then be an irreducible representation of dimension 4, hence $V_{\mathbb{C}} \cong \operatorname{Sym}^3(\mathrm{Std})$ (since representations of $\mathfrak{sl}_{2,\mathbb{C}}$ are classified by their dimension). Let T be a maximal torus of $H_{\mathbb{C}}$ and $t \in X^*(T)$ be a generator of the character group of T.

  The weights of T that occur in the standard representation are $\{-t, t\}$, so the weights of T on $V_{\mathbb{C}}$ are $\{-3t, -t, t, 3t\}$.

  We now use that the map

  $$h_{\mathbb{C}} \circ \mu : \mathbb{G}_{m,\mathbb{C}} \to GL(V_{\mathbb{C}})$$

  factors through the Mumford-Tate group of V, which we write as $\mathbb{G}_{m,\mathbb{C}} \cdot H_{\mathbb{C}}$. As $\mathbb{G}_{m,\mathbb{C}}$ is a torus, the image of such a map is contained in a maximal torus $\tilde{T}$ of $MT(V)$. We can choose T such that $\tilde{T} = \mathbb{G}_{m,\mathbb{C}} \cdot T$. Let $w, \tilde{t}$ be generators for $X^*\left(\tilde{T}\right)$, such that $\tilde{t}$ is sent to t by the map induced on character groups by the inclusion $T \hookrightarrow \tilde{T}$. Then the weights of $\tilde{T}$ on $V_{\mathbb{C}}$ are $cw - 3\tilde{t}, cw - \tilde{t}, cw + \tilde{t}, cw + 3\tilde{t}$ for a certain $c \in \mathbb{Z}$. For the sake of simplicity we identify $X^*(\mathbb{G}_{m,\mathbb{C}}) \cong \mathbb{Z}$ via the character $z \mapsto z$ of $\mathbb{G}_{m,\mathbb{C}}$. Now $\mathbb{G}_{m,\mathbb{C}}$ acts on V through $h_{\mathbb{C}} \circ \mu$, and by definition of μ the weights that occur are -1 and 0; but since $h_{\mathbb{C}} \circ \mu$ factors through $\tilde{T}$, it gives rise to a map in the reverse direction

  $$X^*\left(\tilde{T}\right) \to X^*(\mathbb{G}_{m,\mathbb{C}})$$

  that is not trivial (since $\mathbb{G}_{m,\mathbb{C}}$ acts nontrivially on $V_{\mathbb{C}}$) and $\mathbb{Z}$-linear. We then get a contradiction by observing that the image of this map (that is, $\{-t, 0\}$) should be be symmetric with respect to 0, and it certainly is not.

- $\mathfrak{h}$ cannot be isomorphic $\mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{sl}_{2,\mathbb{C}}$, since the latter has no faithful symplectic 4-dimensional representation. To prove this last assertion, simply note that the faithful representations of $\mathfrak{sl}_{2,\mathbb{C}} \times \mathfrak{sl}_{2,\mathbb{C}}$ are of the form $V_1 \otimes V_2$ with V_1, V_2 faithful representations of the factors. This forces $\dim(V_1) = \dim(V_2) = 2$, which in turn implies that both V_1 and V_2 are isomorphic to the standard representation of $\mathfrak{sl}_{2,\mathbb{C}}$. Since V_1 and V_2 are symplectic, $V_1 \otimes V_2$ is orthogonal (and hence non symplectic, by 2.3.2).

- $\mathfrak{h}$ cannot equal $\mathfrak{g}_2$, since the latter has dimension 28, while the first is contained in $\mathfrak{sp}_{4,\mathbb{C}}$, which has dimension 20.

4.2.2.2 The case of a real quadratic field F

We already know that H is semisimple, thanks to Corollary 3.2.10, and that it is contained in $Sp(V, \psi)$ for a certain bilinear form ψ, thanks to Lemma 3.2.9.

As a special case of Proposition 3.4.2 we get the following Lemma:

Lemma 4.2.3. *There exists a unique F-bilinear form*

$$\varphi : V \times V \to F$$

such that

$$\psi(v_1, v_2) = \operatorname{tr}_{F/\mathbb{Q}}(\varphi(v_1, v_2)) \quad \forall v_1, v_2 \in V.$$

A consequence of this Lemma is that H preserves the F-bilinear form φ: this means at the very least that $H < SL_F(V) := \operatorname{Res}_{F/\mathbb{Q}}(SL_{2,F})$, since a necessary condition for $g \in H$ to preserve φ is $\det_F(g) = 1$. Let U denote $\operatorname{Res}_{F/\mathbb{Q}}(SL_{2,F})$.

Note that - by the very definition of U - its tautological representation in $GL(V)$ becomes isomorphic to the direct sum of two copies of the standard representation of $SL_{2,\mathbb{C}}$ upon extension of scalars to the algebraic closure. Write $V_{\mathbb{C}} \cong \operatorname{Std}_1 \oplus \operatorname{Std}_2$ for this decomposition as $U_{\mathbb{C}}$-representation.

On the other hand, by extending scalars to $\mathbb{C}$ we find that $H_{\mathbb{C}} \hookrightarrow U_{\mathbb{C}} \cong SL_{2,\mathbb{C}} \times SL_{2,\mathbb{C}}$, so by semisimplicity we only have the cases $H_{\mathbb{C}} \cong SL_{2,\mathbb{C}}$ and $H_{\mathbb{C}} \cong SL_{2,\mathbb{C}} \times SL_{2,\mathbb{C}}$, since H is clearly non trivial. Suppose by contradiction that $H_{\mathbb{C}} \cong SL_{2,\mathbb{C}}$. Then from the equality

$$\operatorname{End}(V_{\mathbb{C}})^{H_{\mathbb{C}}} \cong \operatorname{End}(V)^H \otimes_{\mathbb{Q}} \mathbb{C} \cong F \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$$

we see that the only automorphisms of $V_{\mathbb{C}} \cong \mathbb{C}^2 \oplus \mathbb{C}^2$ that commute with $H_{\mathbb{C}}$ are those of the form $\lambda_1 \operatorname{Id}_{\operatorname{Std}_1} \oplus \lambda_2 \operatorname{Id}_{\operatorname{Std}_2}$. This implies that the projections of G on the two factors $SL_{2,\mathbb{C}}$ are surjective, which in turn implies $H \cong SL_2 \times SL_2$ by part (b) of Lemma 3.3.1, since SL_2 admits only one two-dimensional representation, up to isomorphism. We conclude that $H_{\mathbb{C}} \cong SL_{2,\mathbb{C}} \times SL_{2,\mathbb{C}}$ and $H = \operatorname{Res}_{F/\mathbb{Q}}(SL_{2,F})$ (which, in particular, coincides with $L(A)$, the group of endomorphisms of V preserving φ).

4.2.2.3 The quaternion algebra case

Note that D is a skew field of dimension 4 over $\mathbb{Q}$ and that V has, by definition, a structure of D-module. It follows immediately that V is a free D-module of rank 1, hence we can identify $V \cong D$ with its natural action of D by left multiplication. Let $\varphi \in H$. Since the actions of H and D on $V = D$ commute we get

$$\varphi(d) = \varphi(d \cdot 1) = d\varphi(1) \quad \forall d \in D,$$

so φ can be identified to the *right* multiplication by $\varphi(1)$. Moreover, for every pair $\varphi_1, \varphi_2 \in G$ we have

$$(\varphi_1 \circ \varphi_2)(1) = \varphi_2(1)\varphi_1(1),$$

so we can identify H to a subgroup of $(D^{opp})^*$, where D^{opp} is the opposite algebra. Moreover, the discussion in the proof of Lemma 3.2.9 yields

$$H \subseteq \left\{ x | x x^\dagger = 1 \right\},$$

and since $\dagger$ and the standard involution $*$ of D^{opp} are conjugated by an inner automorphism (this is the Skolem-Noether theorem) we get that H is isomorphic to a subgroup of

$$U = \left\{ d \in (D^{opp})^* \left| dd^* = 1 \right. \right\}.$$

Now we know that D is totally undefined, so by extending scalars to $\mathbb{C}$ we find

$$U(\mathbb{C}) = \left\{ d \in (D^{opp} \otimes \mathbb{C})^* \left| dd^* = 1 \right. \right\} = \left\{ d \in M_2(\mathbb{C})^* | dd^* = 1 \right\},$$

and the canonical involution, upon extension of scalars, becomes the adjunction of matrices. It then follows

$$U(\mathbb{C}) = SL_2(\mathbb{C}),$$

so H is contained in a group U that is a $\mathbb{Q}$-form of $SL_2(\mathbb{C})$. But H is semisimple and $SL_{2,\mathbb{C}}$ is simple of rank 1, so $H_\mathbb{C}$ is either SL_2 or trivial. If it were trivial, then H itself would be trivial, which is absurd, because then the endomorphism algebra would be all of $\operatorname{End}_\mathbb{Q}(V)$, which is certainly not the case. We then have (up to isomorphism) $H \subset U$ and $H_\mathbb{C} = U_\mathbb{C}$, so H is isomorphic to U, which in turn clearly agrees with $L(A)$.

4.2.2.4 The CM case

The endomorphism algebra D equals F, a CM field of degree 4 over $\mathbb{Q}$ containing no quadratic imaginary extension of $\mathbb{Q}$.

Lemma 4.2.4. *The absolute Galois group* $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ *acts transitively on the set of embeddings* $F \hookrightarrow \overline{\mathbb{Q}}$. *More precisely, denoting* $\Sigma = \{\sigma, \tau, \overline{\sigma}, \overline{\tau}\}$ *the set of embeddings, there exists* $g \in \operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ *that acts as*

$$\sigma \mapsto \tau \mapsto \overline{\sigma} \mapsto \overline{\tau}.$$

Proof. Suppose first that $F/\mathbb{Q}$ is Galois, and consider the action of $\operatorname{Gal}(F/\mathbb{Q})$, which we write as the quotient $\frac{\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}{\operatorname{Gal}(\overline{\mathbb{Q}}/F)}$, on the set of embeddings given by

$$[g] \cdot \sigma = g \circ \sigma \quad \forall \sigma \in \Sigma, \forall g \in \operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right).$$

This action does factor through $\operatorname{Gal}(F/\mathbb{Q})$ by normality of $F/\mathbb{Q}$. As $[F : \mathbb{Q}] = 4$, $\operatorname{Gal}(F/\mathbb{Q})$ is necessarily Abelian, so the only two possibilities are $\operatorname{Gal}(F/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $\operatorname{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. If we were in the first case, then by Galois theory there would be three quadratic extensions F_1, F_2, F_3 of $\mathbb{Q}$ contained in F. Since F contains no quadratic imaginary extension of $\mathbb{Q}$, F_1, F_2 and F_3 would be real fields, which contradicts the fact that their composite F is a CM field (and hence it does not admit any embedding in $\mathbb{R}$).

It follows that $\operatorname{Gal}(F/\mathbb{Q})$ is cyclic, and it acts as a 4-cycle on the embeddings $F \hookrightarrow \overline{\mathbb{Q}}$: indeed, the stabilizer of each embedding is trivial, whence the orbit of each embedding has length four, so any generator of $\operatorname{Gal}(F/\mathbb{Q})$ acts as a 4-cycle. In particular, a non-trivial element of $\operatorname{Gal}(F/\mathbb{Q})$ has no fixed points.

Now take a $g \in \operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ that projects to a generator $[g]$ of $\operatorname{Gal}(F/\mathbb{Q})$ and let χ denote complex conjugation. If, by contradiction, we had $[g] \cdot \sigma = \overline{\sigma}$, then we would also have $([\chi][g]) \cdot \sigma = \sigma$, so $[\chi][g]$ would have a fixed point, hence it would be the identity of $\operatorname{Gal}(F/\mathbb{Q})$. But this implies $[g] = [\chi] \in \operatorname{Gal}(F/\mathbb{Q})$, and this is an element of order 2, contrary to the assumption that $[g]$ is a generator. Hence $g \cdot \sigma \in \{\tau, \overline{\tau}\}$, and composing with χ if necessary we can assume that g sends σ to τ. The same proof as above shows that τ cannot be sent to $\bar{\tau}$, nor can it be sent to σ, for otherwise $[g]$ would not act as a 4-cycle. This implies $\sigma \mapsto \tau \mapsto \overline{\sigma}$, so the action of $[g]$ is the one we claimed.

On the other hand, suppose that $F/\mathbb{Q}$ is not normal. Let E be the maximal totally real subfield of F. Then F is generated over E by the square root of an element of E, and since $E/\mathbb{Q}$ has degree 2 this shows that F is generated over $\mathbb{Q}$ by a root α_1 of a biquadratic polynomial $p(x)$, with $\alpha_1^2 \in E$ totally negative. Let $\overline{\alpha_1} = -\alpha_1, \alpha_2, \overline{\alpha_2} = -\alpha_2$ be the other roots of this polynomial.

Let N be the normal closure of F in $\overline{\mathbb{Q}}$. Clearly N is generated over $\mathbb{Q}$ by α_1, α_2, so a fortiori is generated over E by α_1, α_2: as both α_1 and α_2 are of degree 2 over E, it follows that $[N : E] \leq 4$ and $[N : \mathbb{Q}] \leq 8$. If $[N : \mathbb{Q}] = 4$ then $F = N$ is normal, contradiction. If, on the other hand, $[N : \mathbb{Q}] = 8$, then the Galois group of $N/\mathbb{Q}$ is isomorphic to the dihedral group on 4 points, since (up to isomorphism) this is the only transitive subgroup of $\mathcal{S}_4$ of order 8.

In this case, take an element σ of order 4 in $\mathrm{Gal}\,(N/\mathbb{Q})$. Then $\sigma(\alpha_1) = \pm\alpha_2$ (if, by contradiction, it sent α_1 to $-\alpha_1$, the it would have order 2). By composing with complex conjugation if necessary we then have $\sigma(\alpha_1) = \alpha_2$, and by the same reasoning $\sigma(\alpha_2) = -\alpha_1$, so σ acts on the roots of $p(x)$ (or, equivalently, on the embeddings $F \hookrightarrow \overline{\mathbb{Q}}$) in the prescribed fashion. As in the previous case, $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ acts through its quotient $\mathrm{Gal}(N/\mathbb{Q})$, hence it contains an element whose action is the one we require.

□

We now proceed on the same lines as in the case of elliptic curves.

Thanks to the general result of Proposition 3.1.10 we know that $Hg(A)$ is commutative, hence Lemma 3.2.9 yields

$$Hg(A) = Z(Hg(A)) \subseteq \left\{x \in \mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_{m,F} | x\overline{x} = 1\right\} =: U_F$$

We now want to study the torus U_F and show that it admits no non-trivial subtori over $\mathbb{Q}$, which in turn will imply that $Hg(A) = U_F$. The (anti)equivalence of categories of Theorem 1.1.1 brings us to consider the character group $X^*\left(U_F\right)$. In order to describe this group, we note that U_F is a sub-object of $T_F := \mathrm{Res}_{F/\mathbb{Q}}\left(\mathbb{G}_{m,F}\right)$, whose character group is free of rank 4: indeed, the general properties of the Weil restriction of scalars imply that

$$\left(T_F\right)_F \cong \mathbb{G}_{m,F}^{[F:\mathbb{Q}]},$$

so

$$\left(T_F\right)_{\mathbb{C}} \cong \left(\left(T_F\right)_F\right)_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}}^4.$$

It is natural to take as generators of $X^*\left(T_{F,\mathbb{C}}\right)$ the characters induced by the four embeddings of F into $\mathbb{C}$, which shall be denoted $\sigma, \tau, \bar{\sigma}, \bar{\tau}$. Let $\Sigma = \{\sigma, \tau, \bar{\sigma}, \bar{\tau}\}$. The subtorus U_F is defined by $x\bar{x} = 1$, so it corresponds to the quotient

$$\mathrm{Hom}\left(U_{F,\mathbb{C}}, \mathbb{G}_{m,\mathbb{C}}\right) = \frac{\mathrm{Hom}\left(T_{F,\mathbb{C}}, \mathbb{G}_{m,\mathbb{C}}\right)}{\{\text{homomorphisms trivial on those } x \text{ such that } x\bar{x} = 1\}}$$

$$\cong \frac{\bigoplus_{\gamma\in\Sigma} \mathbb{Z}\gamma}{\mathbb{Z}\left(\sigma + \bar{\sigma}\right) \oplus \mathbb{Z}\left(\tau + \bar{\tau}\right)}.$$

We can now show that this character group admits no non-trivial quotients in the category of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$-modules. Lemma 4.2.4 ensures the existence of an element $g \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ acting on Σ by sending $\sigma \mapsto \tau \mapsto \overline{\sigma} \mapsto \overline{\tau}$. Quotients of $X^*(U_F)$ correspond to quotients of $X^*\left(T_F\right)$ by $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$-submodules containing $N := \mathbb{Z}\left(\sigma + \overline{\sigma}\right) \oplus \mathbb{Z}\left(\tau + \overline{\tau}\right)$. Let M be any such submodule, and suppose $M \neq N$. Every element of M can be represented as $m = a\sigma + b\tau + c(\sigma + \overline{\sigma}) + d(\tau + \overline{\tau})$. To say that M is strictly larger than N is to say that there exists a certain m with $(a,b) \neq (0,0)$. Without loss of generality, by adding suitable multiples of $\sigma+\overline{\sigma}$ and $\tau+\overline{\tau}$, we can take $(c,d)=(0,0)$. But M is a $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$-module, so M also contains $g \cdot m = a\tau + b\overline{\sigma}$, therefore it contains $m' := g \cdot m - b\left(\sigma + \overline{\sigma}\right) = a\tau - b\sigma$ and

$$am - bm' = (a^2+b^2)\sigma.$$

The orbit of this last element under the action of g generates a free submodule of rank 4, hence the quotient $X^*(T_F)/M$ is a finite group. Since tori over $\mathbb{Q}$ correspond to *free* abelian groups (with a continuous action of $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$), and the only group that is at the same time finite and free is the trivial one, we conclude that T_F does not admit any non-trivial subtorus defined over $\mathbb{Q}$, as claimed.

Finally, $L(A)$ commutes with $\operatorname{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$, so it is contained in T_F, and in fact it is also a subgroup of U_F, since it preserves the form induced by the polarization. Now $U_F = Hg(A)$, so $L(A)$, U_F and $Hg(A)$ must all coincide.

CHAPTER 5

Three conjectures

We collect here the statements of three famous, closely related and (as yet) unproven conjectures regarding algebraic varieties, together with a list of results that may be considered both as evidence and motivation for such conjectures.

5.1 Hodge conjecture

Let X be a complex Abelian variety (or, in fact, any complex Kähler manifold). A famous theorem of Hodge asserts the existence of a decomposition

$$H^n(X, \mathbb{C}) \cong \bigoplus_{p+q=n} H^{p,q}(X),$$

where $H^{p,q}(X)$ is the space of cohomology classes that can be represented by harmonic forms of type (p,q); such a decomposition is therefore compatible with the cup product.

Definition 5.1.1. Let $i : Z \hookrightarrow X$ be an (analytic) subvariety of complex codimension k. Since the singular locus has complex codimension $\geq k+1$ (hence real codimension at least $2k+2$), the integration map

$$\begin{array}{ccc} H^{2n-2k}_{deRham}(A) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int_Z \omega \end{array}$$

is well-defined (via Stokes' theorem) and linear, so by Poincaré duality there exists a cohomology class $[Z] \in H^{2k}_{deRham}(X)$, called the **Poincaré dual of [Z]**, such that

$$\int_Z \omega = \int_X \omega \wedge [Z]$$

for every closed form ω.

Let now $L^k(X)$ be the free Abelian group on the subvarieties of complex codimension k of L. The **cycle map** is defined to be

$$\begin{array}{rccc} cyc: & L^k(X) & \to & H^{2k}_{deRham}(X) \\ & \sum c_i Z_i & \mapsto & \sum c_i [Z_i] \end{array}$$

Remark 5.1.2. It can be shown that the cycle map respects rational equivalence ([Ful84], Proposition 19.1.1), so there is a notion of cycle map $A^k(X) \to H^{2k}_{deRham}(X)$, where $A^k(X)$ is the k-th Chow group, the quotient by rational equivalence of the free Abelian group on subvarieties of codimension k.

Proposition 5.1.3. *Let $i : Z \hookrightarrow X$ be a complex submanifold of (complex) dimension k. Then $[Z]$ lies in $H^{n-k,n-k}(X)$.*

Proof. For every point $z \in Z$ we can choose a neighborhood U_z of z in X and local coordinates $x_1^{(z)}, \cdots, x_n^{(z)}$ such that

$$Z \cap U_z = \left\{ \left(x_1^{(z)}, \cdots, x_n^{(z)} \right) | x_i^{(z)} = 0 \text{ for } i = k+1, \cdots, n \right\}.$$

By compactness of Z (that is a closed subspace of the compact manifold X) we can then extract from the collection of the U_z's a finite open cover $\mathcal{U} = \{U_j\}_{j=1,\cdots,m}$ of Z. Fix a partition of unity ψ_j subordinated to $\mathcal{U}$ and let α be a differential form of type $(p, 2k-p) =: (p, q)$ defined on the whole of X.

We want to show that unless $p = k$ we have $\int_Z i^*\alpha = 0$.

Using partitions of unity we reduce to a local problem: indeed, for each index j, the form $\psi_j\alpha$ has the same type as α, and it is supported in U_j. If we knew the claim to show holds for forms supported in one of the U_j's we would then have

$$\int_Z i^*\alpha = \int_Z i^* \left(\sum_{j \in J} \psi_j \alpha \right) = \sum_{j \in J} \int_Z i^* (\psi_j \alpha) = 0.$$

We can therefore suppose α to be supported in U_j, where we have local coordinates $x_i := x_i^{(j)}$. α can then be written as a sum of terms of the form $\varphi dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge \overline{dx^{l_1}} \wedge \cdots \wedge \overline{dx^{l_q}}$, where φ is a smooth function and $i_1 < \cdots < i_p$. Obviously we can assume this sum to be made of just one term. Now suppose $p > k$. Then $i_p > k$, so by our choice of local coordinates in U_j we have $x^{i_p} \equiv 0$ on Z, which clearly implies

$$i^*\alpha = (\varphi \circ i)\, i^*dx^{i_1} \wedge \cdots \wedge i^*dx^{i_p} \wedge i^*\overline{dx^{l_1}} \wedge \cdots \wedge i^*\overline{dx^{l_q}} =$$

$$(\varphi \circ i)\, i^*dx^{i_1} \wedge \cdots \wedge 0 \wedge i^*\overline{dx^{l_1}} \wedge \cdots \wedge i^*\overline{dx^{l_q}} = 0,$$

so a fortiori the integral of $i^*\alpha$ on Z is zero. The same argument shows that if $q > k$, then $l_q > k$ and $i^*(\alpha) = 0$.

On the other hand, by definition of the Poincaré dual,

$$\int_Z i^*\alpha = \int_X \alpha \wedge [Z],$$

and we have just shown that this expression is zero for every α not of type (k, k). Write $[Z] = \sum_{p+q=2n-2k} [Z]^{p,q}$ for the Hodge decomposition of $[Z]$.

For any non-zero smooth closed form ω of type (p, q) we can find a smooth form χ of type $(n-p, n-q)$ such that $\int_X \omega \wedge \chi \neq 0$ (it is enough to do this locally, where it is clear). Applying this to $[Z]^{(p,q)}$ we find that all these form vanish but for $[Z]^{(n-k,n-k)}$, so $[Z]$ lies in $H^{n-k,n-k}$ as claimed. □

Definition 5.1.4. Let X be a complex projective manifold. The elements in the intersection

$$\mathscr{B}^k(X) := H^{k,k}(X) \cap H^{2k}(X, \mathbb{Q})$$

are called the **Hodge classes of degree** $2k$.

From now on we will denote $\mathscr{B}^\bullet(X) := \bigoplus_{k\geq 0} \mathscr{B}^k(X)$ the Hodge ring of X, and $\mathscr{D}^\bullet(X) = \bigoplus_{k\geq 0} \mathscr{D}^k(X)$ the $\mathbb{Q}$-subalgebra of $\mathscr{B}^\bullet(X)$ generated by divisor classes.

Remark 5.1.5. We have here a slight conflict of notation with our previous definition of Hodge classes, but this can be explained as follows: the space $H^k(X, \mathbb{Q})$ has a Hodge decomposition coming from the Hodge decomposition of $H^k(X, \mathbb{C})$ through the canonical isomorphism $H^k(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^k(X, \mathbb{C})$.

The natural Hodge structure on $H^k(X, \mathbb{C})$ is pure of weight $-2k$, while we defined Hodge classes exclusively for structures of weight zero; in order to connect the two definitions, note that Hodge classes of degree $2k$ are simply the Hodge classes (as previously defined) in $H^{2k}(X, \mathbb{Q}) \otimes \mathbb{Q}(-k)$, where $\mathbb{Q}(-k)$ is the Tate structure of pure weight $2k$.

Conjecture 5.1.6 (The Hodge Conjecture)**.** *Let X be a complex projective manifold of complex dimension n. Then, for every $k = 0, \ldots, n$, the image of the cycle map $cyc : A^k(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2k}(X, \mathbb{Q})$ equals the set $\mathscr{B}^k(X)$ of Hodge classes of degree $2k$.*

Remark 5.1.7. For $k = 0$ and $k = 2n$, the stronger equality

$$H^{k,k}(X) \cap H^{2k}(X, \mathbb{Z}) = \text{Image}\left(cyc : A^k(X) \to H^{2k}(X, \mathbb{Z})\right)$$

trivially holds because of dimension considerations. We can assume X to be connected, so both $H^0(X, \mathbb{Z})$ and $H^{2n}(X, \mathbb{Z})$ are free modules of rank one (the latter being generated by the fundamental class of X). The Poincaré dual of the whole manifold is clearly the constant function 1, that generates $H^0(X, \mathbb{Z})$ since X is connected, and on the other hand the Poincaré dual of a point p is the fundamental class of X: indeed, a closed 0-form is a constant function $\varphi \equiv \varphi(p)$, and we trivially have

$$\int_p \varphi = \varphi(p) = \varphi(p) \cdot \int_X [X] = \int_X \varphi \wedge [X].$$

This shows that the image of the cycle map contains generators for both $H^0(X, \mathbb{Z})$ and $H^n(X, \mathbb{Z})$, so *cyc* is surjective.

A powerful tool to study all sorts of questions connected to the cohomology ring of manifolds is the so-called Hard Lefschetz Theorem, that we state here for completeness:

Theorem 5.1.8 (Hard Lefschetz Theorem)**.** *Let X be a n-dimensional non-singular complex projective variety. Fix an embedding $X \hookrightarrow \mathbb{P}^n\mathbb{C}$. Let ω be the cohomology class in $H^2(X, \mathbb{Z})$ of any hyperplane divisor on X. Then, for every $k \geq 0$, taking the k-fold wedge product with ω gives an isomorphism*

$$H^{n-k}(X, \mathbb{Z}) \cong H^{n+k}(X, \mathbb{Z}).$$

5.1.1 The Lefschetz theorem on $(1,1)$ classes

The first interesting case of the Hodge conjecture (and, to date, the only one having been completely solved) is $k = 1$; the Lefschetz theorem on $(1,1)$ classes is an even stronger statement ([GH94], pag. 163):

Theorem 5.1.9. *We have the equality $H^2(X, \mathbb{Z}) \cap H^{1,1}(X) = \{[Z] | Z \text{ divisor}\}$*

Remark 5.1.10. Note that this is stronger than the Hodge conjecture as formulated above, since there we allow the coefficients to vary in $\mathbb{Q}$ (and by so doing, we lose information about the torsion part of the various cohomology groups). In fact, the original statement of the Hodge conjecture was that every cohomology class in $H^{2k}(X, \mathbb{Z}) \cap H^{k,k}(X)$ is the cohomology class of an algebraic cycle with integral coefficients on X, but this is now known to be false (the first counterexample appeared in [AH61]).

Proof. Consider the exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O}_X \xrightarrow{\exp 2\pi i \cdot} \mathcal{O}_X^* \to 0$$

and the segment of the long exact cohomology sequence defining the first Chern class map,

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H_2(X, \mathbb{Z}) \xrightarrow{i_*} H_2(X, \mathcal{O}_X).$$

It can be shown (see for example [GH94], page 141) that the cycle map agrees with the first Chern class on divisors, so we are left with showing that c_1 surjects onto the space of (integral) Hodge classes. By exactness, this is equivalent to showing that i_* vanishes on this space; but we already know that every Hodge class lies in $H^{1,1}(X)$ (by Proposition 5.1.3), so it is enough to show that i_* vanishes on $H^{1,1}(X)$. We can factor the injection i as

$$\mathbb{Z} \xrightarrow{i_1} \mathbb{C} \xrightarrow{i_2} \mathcal{O}_X,$$

so it is enough to show that the induced map $(i_2)_* : H^2(X, \mathbb{C}) \to H^2(X, \mathcal{O}_X)$ is zero. Using the Hodge decomposition we can write

$$H^2(X, \mathbb{C}) \cong \bigoplus_{p+q=2} H^{p,q}(X);$$

Hodge theory also identifies $H^{0,2}(X)$ with $H^2(X, \mathcal{O}_X)$, in such a way that $(i_2)_*$ simply becomes the natural projection

$$(i_2)_* : \bigoplus_{p+q=2} H^{p,q}(X) \twoheadrightarrow H^{0,2}(X),$$

which is clearly zero on $H^{1,1}(X)$. □

Remark 5.1.11. Combining the above Lefschetz theorem with the aforementioned Theorem 5.1.8 we see that the Hodge conjecture holds (even in its integral form) for every Abelian variety of dimension at most 3.

5.1.2 A theorem of Hazama and Kumar Murty

In view of the Lefschetz theorem on $(1,1)$ classes, there is a simple case in which the Hodge conjecture holds, namely if the Hodge ring is generated in degree 2: in this case, all Hodge classes are combinations of products of Hodge classes of degree 2, which in turn are known to be divisor classes. This leads naturally to the following definition:

Definition 5.1.12. An **exceptional Hodge class** is an element of $\mathscr{B}^* \setminus \mathscr{D}^*$.

In what follows we are going to use repeatedly the following criterion, due (independently) to F. Hazama ([Haz92]) and Kumar Murty ([Mur84]), that gives necessary and sufficient conditions for the favorable equality $\mathscr{D}^*(A^n) = \mathscr{B}^*(A^n)$ to hold for all the powers of an Abelian variety A.

Theorem 5.1.13. *Keeping all of the above notation, we have $\mathscr{B}^*(A^n) = \mathscr{D}^*(A^n)$ for all $n \geq 1$ if and only if the following two conditions hold:*

- *A has no simple factor of type III;*
- *the Hodge group of A equals the Lefschetz group of A.*

5.2 Tate conjecture

In a much more arithmetic context, Tate asked if a Hodge-like decomposition could hold for the ℓ-adic étale cohomology of a proper and smooth variety (with some technical assumptions on the field of definition): as proved by Tate and Raynaud for Abelian varieties and by Faltings in full generality ([Fal88]), such a decomposition exists, and it shares many properties with the geometric (Hodge) case.

There is also an abstract definition, which will be given later in this section, of modules "of Hodge-Tate type", the ℓ-adic analogue of abstract Hodge structures.

A few references for this section are [Tat66], [Ser67] and [Ser79].

Before getting started on the Hodge-Tate decomposition itself let us recall a few basic facts about the Tate module of an Abelian variety. Let K be a number field with absolute Galois group $\Gamma_K = \operatorname{Gal}\left(\overline{K}/K\right)$ and A a g-dimensional Abelian variety defined over K. For every prime ℓ, the $\overline{K}$-valued points of A form a ℓ-divisible Abelian group, and we can consider

$$T_p(A) = \varprojlim A\left[\ell^n\right],$$

where $A\left[\ell^n\right]$ is the set of ℓ^n-torsion points of $A_{\overline{K}}$. $T_\ell(A)$ is called the ℓ-adic Tate module of A, and it comes equipped with a continuous action of $\operatorname{Gal}\left(\overline{K}/K\right)$ coming from its natural action on $A\left[\ell^n\right]$ for every n.

Classical results on abelian varieties show that (since K has characteristic zero) $T_\ell(A)$ is a free module over $\mathbb{Z}_\ell$ of rank $2g$ and that it can be identified canonically to the dual (as Galois modules) of the étale cohomology group $H^1_{ét}(A \times_K \overline{K}, \mathbb{Z}_\ell)$. It will also be convenient to introduce $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, which is clearly a $\mathbb{Q}_\ell$-vector space of dimension $2g$.

Finally, note that the comparison isomorphism between étale and classical cohomology yields canonical identifications $V_\ell \cong H_1\left(A, \mathbb{Q}\right) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$.

We can now turn to the description of the Hodge-Tate decomposition. Choose a place w of K of good reduction for A and of residue characteristic ℓ, a place $\overline{w}$ of $\overline{K}$ above w and let $I_{\overline{w}}$ the absolute inertia group of $\overline{w}$. Fix furthermore a completion $\mathbf{C}$ of $\overline{\mathbb{Q}_\ell}$. There is a natural action of $I_{\overline{w}}$ on V_ℓ, which we can extend semi-linearly to $\overline{V_\ell} := V_\ell \otimes_{\mathbb{Q}_\ell} \mathbf{C}$ via the formula

$${}^{\sigma}\left(v \otimes c\right) = {}^{\sigma}v \otimes {}^{\sigma}c \quad \forall \sigma \in I_{\overline{w}},$$

which makes sense because every automorphism of an ℓ-adic field is automatically continuous, and so the action extends (continuously) to $\mathbf{C}$.

Set

$$\overline{V_\ell}\left\{i\right\} = \left\{v \in \overline{V_\ell} | \sigma v = \chi_\ell(\sigma)^i v \, \forall \sigma \in I_{\overline{q}}\right\}$$

and $\overline{V_\ell}(i) := \overline{V_\ell}\{i\} \otimes \mathbf{C}$. The inclusions $\overline{V_\ell}\{i\} \hookrightarrow \overline{V_\ell}$ extend, by a theorem of Tate-Serre ([Ser67], Proposition 4), to an injection

$$\bigoplus_i \overline{V_\ell}(i) \hookrightarrow \overline{V}_\ell.$$

The key result Tate and Raynaud were able to prove is that this is in fact an isomorphism, so that we have a Hodge-style decomposition

$$H^1_{ét}\left(A_{\overline{K}}, \mathbb{Q}_\ell\right) \cong \overline{V_\ell}(0) \oplus \overline{V_\ell}(1);$$

moreover, there are natural identifications

$$\overline{V_\ell}(0) \cong \mathrm{Lie}\left(A_{\mathbf{C}}\right), \quad \overline{V_\ell}(1) \cong \mathrm{Lie}\left(A^\vee_{\mathbf{C}}\right)(1),$$

where $A^\vee$ is the dual Abelian variety. We record it here as a theorem:

Theorem 5.2.1. *If A has good reduction at a place w of residue characteristic ℓ, then*

$$H^1_{ét}\left(A_{\overline{K}}, \mathbb{Q}_\ell\right) \cong \mathrm{Lie}\left(A_{\mathbf{C}}\right) \oplus \mathrm{Lie}\left(A^\vee_{\mathbf{C}}\right)(1)$$

Remark 5.2.2. More generally, let K be an ℓ-adic field and $V_{\mathbf{C}}$ be a finite-dimensional $\Gamma_K := \mathrm{Gal}\left(\overline{K}/K\right)$-module over $\mathbf{C}$. Let

$$V_{\mathbf{C}}\{n\} := \left\{x \in V_{\mathbf{C}} | g \cdot x = \chi(g)^{-n} x \, \forall g \in \Gamma_K\right\}$$

(notice that these are only K-, and not $\mathbf{C}$-, subspaces of $V_{\mathbf{C}}$) and

$$V_{\mathbf{C}}(n) := V_{\mathbf{C}}\{n\} \otimes_K \mathbf{C}(-n).$$

The aforementioned result of Tate and Serre ([Ser67], Proposition 4) implies the existence of an injective, canonical morphism of Γ_K-modules

$$\bigoplus_{n\in\mathbb{Z}} V_{\mathbf{C}}(n) = \bigoplus_{n\in\mathbb{Z}} \left(V_{\mathbf{C}}\{n\} \otimes \mathbf{C}(-n)\right) \hookrightarrow V_{\mathbf{C}},$$

and a module is said to be of **Hodge-Tate** type if this is an isomorphism. If W is a Γ_K-module defined over $\mathbb{Q}_\ell$, then W is said to be of Hodge-Tate type if $W \otimes_{\mathbb{Q}_\ell} \mathbf{C}$ is Hodge-Tate in the above sense.

Tate-Raynaud's theorem can then be restated by saying that the Galois module afforded by the Tate module of an Abelian variety (of good reduction at a place dividing ℓ) is of Hodge-Tate type.

Building on this result, Tate also formulated an analogue of the Hodge conjecture for ℓ-adic cohomology, proposing that a connection should exist between the algebraic cycles on smooth varieties and the Galois modules afforded by their étale cohomology groups.

Let X be a smooth projective variety over a field k (finitely-generated over its prime field). Fix a separable closure $\overline{k}$ of k, a rational prime ℓ and let Γ_k be the absolute Galois group of k. Let $H^\bullet(X)$ be the ℓ-adic cohomology of the base extension of X to $\overline{k}$, endowed with its natural Γ_k-module structure, and consider $H^{2i}(X)(i)$, the i-fold Tate twists of the even-numbered cohomology groups. Write Γ_ℓ for the image of Γ_k inside $GL\left(H^{2i}(X)(i)\right)$ (which is an ℓ-adic compact Lie group) and $\mathfrak{g}_\ell$ for its Lie algebra. With this notation, the Tate conjecture reads

Conjecture 5.2.3 (Tate Conjecture)**.** *The $\mathfrak{g}_\ell$-invariants inside $H^{2i}(X)(i)$ are generated (as $\mathbb{Q}_\ell$-vector space) by the classes of algebraic cycles.*

As for the structure of $\mathfrak{g}_\ell$ we have the following result:

Theorem 5.2.4 (Bogomolov, [Bog80])**.** *$\mathfrak{g}_\ell$ is algebraic (that is, $\mathfrak{g}_\ell$ is also the Lie algebra of G_ℓ) and it contains the homotheties.*

As a corollary, we can always write $\mathfrak{g}_\ell = \mathbb{Q}_\ell \operatorname{id} \oplus \mathfrak{h}_\ell$ where $\mathfrak{h}_\ell$ - the set of elements of trace zero - plays a role analogue to that of the Hodge algebra in the geometric case (in fact, $\mathfrak{h}_\ell = \operatorname{Lie}(H_\ell)$). Note that passing from an ℓ-adic Lie group to an open subgroup does not change its Lie algebra, so $\mathfrak{g}_\ell$ does not change under finite extensions of K (whereas G_ℓ can change); therefore, we can always enlarge K so that all the endomorphisms of A are defined over K without altering $\mathfrak{g}_\ell$, and we will always work under this assumption.

Also, note that G_ℓ is not necessarily connected, but replacing K by a finite extension we can (and will) assume that this is the case; moreover, this finite extension can be chosen independently of ℓ (see Theorem 3.3.2 in [Ser85] and Section 3.b of [Chi90]).

Remark 5.2.5. An equivalent way of stating the Tate conjecture is through the notion of **Tate classes**, that is to say, cohomology classes c in $H^{2i}_{ét}\left(X_{\overline{k}}, \mathbb{Q}_\ell\right)$ such that the action of every $\sigma \in \Gamma_k$ is given by $\sigma \cdot c = \chi(\sigma)^{-i} c$, where χ denotes the ℓ-adic cyclotomic character.

Note that the conjecture can be formulated for arbitrary smooth projective varieties, but we will only discuss the setting of Abelian varieties. In this case, if we set $V = H_1(A, \mathbb{Q})$ and $V_\ell = V \otimes \mathbb{Q}_\ell$, we have, as discussed at the beginning of the section, a canonical identification $V_\ell \cong V_\ell(A)$; moreover, for every n we have a natural isomorphism of $\operatorname{Gal}(\overline{K}/K)$-modules

$$H^n_{ét}\left(A_{\overline{K}}, \mathbb{Q}_\ell\right) \cong \operatorname{Hom}\left(\bigwedge^n V_\ell, \mathbb{Q}_\ell\right),$$

so we can basically reduce all of our study to the Galois module V_ℓ.

Finally, note that - in complete analogy to the geometric case - if E is a subfield of $\operatorname{End}^0(A)$, then V_ℓ acquires the structure of a free $E_\ell := E \otimes$

$\mathbb{Q}_\ell$-module of rank $\frac{2\dim(A)}{[E:\mathbb{Q}]}$. If $\varphi : V \times V \to \mathbb{Q}$ is the Riemann form of a polarization, then φ induces a form φ_ℓ on V_ℓ, and our arguments on bilinear forms now translate in the statement that there exists a E_ℓ-bilinear (resp. Hermitian, in the CM case) form on V_ℓ that is preserved by the action of $\mathfrak{h}_\ell$. With obvious notation, we will write $\mathfrak{h}_\ell \subseteq \mathfrak{sp}_{E_\ell}(V_\ell, \psi_\ell)$ (resp. $\mathfrak{h}_\ell \subseteq \mathfrak{u}_{E_\ell}(V_\ell, \psi_\ell)$ in the CM case).

Definition 5.2.6. The **Tate ring** of a variety A (defined over a number field K) is $\mathscr{T}_\ell^\bullet(A) = \bigoplus_i \mathscr{T}_\ell^i$, where $\mathscr{T}_\ell^i$ is the space of Tate classes in $H^{2i}_{ét}\left(A_{\overline{K}}, \mathbb{Q}_\ell\right)$

Faltings proved, in his famous paper [Fal83], the analogue of the Lefschetz theorem on (1,1) classes (Theorem 5.1.9), namely that every Tate class in the second étale cohomology group is a linear combination of divisor classes with coefficients in $\mathbb{Q}_\ell$. From now on, let $\mathscr{D}_\ell^\bullet(A)$ be the sub-algebra of $\mathscr{T}_\ell^\bullet(A)$ generated by divisor classes. It follows from Faltings' result that the Tate conjecture, pretty much like in the geometric case, holds in every codimension as soon as the Tate ring is generated by divisor classes; minor modifications of the proof of Theorem 5.1.13 yield an ℓ-adic criterion for this to hold not only for A but for its powers as well,

Theorem 5.2.7. *Necessary and sufficient conditions for the equality*

$$\mathscr{T}_\ell^\bullet(A^n) = \mathscr{D}_\ell^\bullet(A^n)$$

to hold for every $n \geq 1$ are

- *A has no simple factor of type III*
- *$\mathfrak{h}_\ell = \mathfrak{sp}_{\mathrm{End}^0(A)\otimes\mathbb{Q}_\ell}(V_\ell, \varphi_\ell)$, where φ_ℓ is the alternating bilinear form induced on V_ℓ by the choice of a polarization*

Remark 5.2.8. The analogous of the Hard Lefschetz Theorem was proved by Deligne in [Del80], so the Tate conjecture holds for every Abelian variety of dimension at most 3.

Hodge-Tate modules arising from Abelian varieties thus bear a great resemblance to Hodge structures, and it is easy to conjecture results analogous to those of Chapter 3: in many cases, these are known to be true, and we will see in the next section that this has led to the formulation of a general conjecture relating the geometric Hodge decomposition and the arithmetic Hodge-Tate structure.

5.3 Mumford-Tate conjecture

Let A be an Abelian variety over a number field K. Fix a prime number ℓ and let $T_\ell(A)$ be the ℓ-adic Tate module of A. Let furthermore $V_\ell(A) = T_\ell(A) \otimes \mathbb{Q}_\ell$ and

$$\rho_\ell : \operatorname{Gal}\left(\overline{K}/K\right) \to \operatorname{Aut}\left(T_\ell(A)\right) \hookrightarrow \operatorname{GL}\left(V_\ell(A)\right)$$

be the associated Galois representation. The image of ρ_ℓ is an ℓ-adic Lie subgroup of $\operatorname{GL}(V_\ell(A))$; let G_ℓ be its Zariski closure, i.e. the smallest algebraic subgroup of $\operatorname{GL}(V_\ell(A))$ containing it. By analogy with the geometric case, we also introduce the ℓ-adic version of the Hodge group, H_ℓ, which we define as the connected component of the identity of $G_\ell \cap SL(V_\ell(A))$. Let furthermore $\mathfrak{g}_\ell$ the Lie algebra of $\operatorname{Im}(\rho_\ell)$.

A fundamental problem is to determine the Lie algebras $\mathfrak{g}_\ell$ for varying ℓ; a conjectural answer to this question is given by the **Mumford-Tate conjecture**.

Conjecture 5.3.1 (Mumford-Tate Conjecture)**.** *Fix an embedding $K \hookrightarrow \mathbb{C}$, so that A acquires the structure of an Abelian variety over $\mathbb{C}$. Let G be the Mumford-Tate group of A, $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}_\ell$ be as above.*

Then for every rational prime ℓ the equality $\mathfrak{g}_\ell = \mathfrak{g} \otimes \mathbb{Q}_\ell$ holds inside $\mathfrak{gl}(V_\ell)$.

Remark 5.3.2. In the presence of the Mumford-Tate conjecture, the Hodge and Tate conjectures are equivalent.

Many partial (but still very deep) results have been established that go in the direction of the above conjecture. We state here, without proofs, the main ones.

Theorem 5.3.3 (Faltings, [Fal83])**.** *The Lie algebra $\mathfrak{g}_\ell$ is reductive and its centralizer in $\operatorname{End}(V_\ell)$ equals $\operatorname{End}^0(A) \otimes \mathbb{Q}_\ell$.*

Theorem 5.3.4 (Sen, [Sen73]; cf. Proposition 3.1.2)**.** *Let V be a module of Hodge-Tate type over $\mathbb{Q}_\ell$ and $\Phi \in \operatorname{End}_{\mathbf{C}}(V_{\mathbf{C}})$ be the element such that the restriction of Φ to $V(i)$ is multiplication by i.*

Then the Lie algebra of G_ℓ is the smallest $\mathbb{Q}_\ell$-subspace $\mathfrak{g}$ of $\operatorname{End}_{\mathbb{Q}_\ell}(V)$ such that $\mathfrak{g} \otimes_{\mathbb{Q}_\ell} \mathbf{C}$ contains Φ.

Also, reasoning along the same lines of Corollary 3.2.10 it is possible to show that if A has no simple factor of type IV, then $\mathfrak{h}_\ell$ is the semi-simple part of $\mathfrak{g}_\ell$.

Another key tool for working in the ℓ-adic setting is the following theorem of Serre

Theorem 5.3.5 ([Ser85] 2.2.4)**.** *The rank of $\mathfrak{g}_\ell$ is independent of ℓ.*

Moreover, Deligne has proved 'one half' of the conjecture, namely

Theorem 5.3.6 ([DMOS82], I, Proposition 6.2)**.** *For every rational prime ℓ the inclusion $G_\ell \subseteq MT(A) \otimes \mathbb{Q}_\ell$ holds.*

As a consequence of the above results it can be shown that the Mumford-Tate conjecture (for a fixed A) holds for every prime whenever it holds for at least one prime ([LP95], Theorem 4.3).

Finally, we note here that the Mumford-Tate conjecture is known for CM varieties: this has been pointed out by Ribet in [Rib90] and is essentially due to Taniyama and Shimura, see [ST61].

CHAPTER 6

Computing Mumford-Tate groups

We are now in a position to prove properties of the Mumford-Tate groups that will lead, in some particular cases, to its precise determination.

After having fixed our notation, in 6.2 we prove that the tautological representations of Mumford-Tate groups are defined by minuscule weights. We follow here the approach of Zarhin ([Zar85], Theorem 0.5.1), but note that the result was already known to Serre ([Ser79], Proposition 7), who had given a different proof.

As a first application, in 6.3 we recount the proof of a theorem, due to Pink ([Pin98], Theorem 5.14), on varieties with endomorphism ring $\mathbb{Z}$ and whose dimension lies outside an exceptional set which can be described explicitly.

The next section is then dedicated to a more concrete description of the Lefschetz group, which will allow us to identify, in 6.5 and 6.6, some cases where the equality between the Hodge and Lefschetz groups holds.

6.1 Notation

We now fix the notation we will constantly be using from now on; we want it to be tailored in a way that allows us to treat geometric and ℓ-adic questions almost at the same time. We will therefore use the symbols $\mathfrak{g}, \mathfrak{h}, V, \mathbf{Q}, \mathbf{C}, E, \varphi, \psi$ to mean

- in the geometric case,

$$\mathfrak{g} = \mathrm{Lie}(MT(A)), H = Hg(A), \mathfrak{h} = \mathrm{Lie}(Hg(A)), V = H_1(A, \mathbb{Q}),$$

$$\mathbf{Q} = \mathbb{Q}, \mathbf{C} = \mathbb{C}, D = \mathrm{End}^0(A),$$

φ the bilinear form induced on V by the choice of a polarization,

ψ the $D-$bilinear or skew-Hermitian form inducing φ by taking traces;

- in the ℓ-adic case,

$$\mathfrak{g} = \mathrm{Lie}(G_\ell(A)), H = H_\ell(A), \mathfrak{h} = \mathrm{Lie}(H_\ell(A)), V = V_\ell,$$

$$\mathbf{Q} = \mathbb{Q}_\ell, \mathbf{C} = \hat{\overline{\mathbb{Q}_\ell}}, D = \mathrm{End}^0(A) \otimes \mathbb{Q}_\ell$$

and φ, ψ as above (with the appropriate meaning of V, D).

6.2 Mumford-Tate representations are minuscule

Let G be a reductive group over a field k, and let K be an algebraically closed field of characteristic zero containing k. Write G_K as almost-direct product $G_0 \cdot G_1 \cdots G_n$ of its center G_0 and its simple factors $G_1, \ldots, G_n$. Let $\mathfrak{g}$ be the Lie algebra of G_K^{der} and $\mathfrak{g}_i, i = 1, \ldots, n$ be the (simple) Lie algebra of G_i; let furthermore $\mathfrak{c}$ be Lie $(G_{0,K})$, which is canonically isomorphic to $X_*(G_{0,K}) \otimes K$. We clearly have $\mathrm{Lie}(G_K) \cong \mathfrak{c} \times \mathfrak{g}$.

Write $G_i' := G_K / \left(G_0 G_1 \cdots \hat{G}_i \cdots G_n \right)$, where the symbol $\hat{G}_i$ means that the factor G_i is omitted in the product. Let p_i be the canonical projection $G_K \twoheadrightarrow G_i'$. Let furthermore $\gamma : \mathbb{G}_{m,K} \to G_K$ be a cocharacter of G. Applying the functor Lie to γ we get a homomorphism of Lie algebras over K,

$$\mathrm{Lie}(\gamma) : \quad \mathrm{Lie}\left(\mathbb{G}_{m,K}\right) \quad \cong \quad K \quad \to \quad \mathfrak{c} \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n \quad \cong \quad \mathrm{Lie}(G_K).$$

Let us consider $\mathrm{Lie}(\gamma)$ as linear map from K to $\mathfrak{c} \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ and write the image of x as $(\mathfrak{l}_0(x), \mathfrak{l}_1(x), \ldots, \mathfrak{l}_n(x))$.

As $\mathbb{G}_{m,K}$ is a torus, we can choose a maximal torus of G_K containing the image of γ; such a choice induces a choice of Cartan subalgebras $\mathfrak{h}_i$ of $\mathfrak{g}_i$ such that $\mathrm{Lie}(\gamma)$ factors through $\mathfrak{c} \times \mathfrak{h}_1 \times \cdots \times \mathfrak{h}_n$. Moreover, we can choose maximal tori $H_i \subset G_i'$ such that $\mathrm{Lie}(H_i) \cong \mathfrak{h}_i$. Let R_i be the root systems associated to the Cartan subalgebras $\mathfrak{h}_i$ and R be the root system of $\mathfrak{g}$.

Dualizing the K-linear map $\mathrm{Lie}(\gamma)$ we get a morphism

$$\begin{array}{rccc} \varphi_\gamma : & \mathfrak{c}^\vee \times \mathfrak{h}_1^\vee \times \cdots \times \mathfrak{h}_n^\vee & \to & K \\ & (\lambda_0, \lambda_1, \cdots, \lambda_n) & \mapsto & \lambda_0 \circ \mathfrak{l}_0(1) + \lambda_1 \circ \mathfrak{l}_1(1) + \cdots + \lambda_n \circ \mathfrak{l}_n(1). \end{array}$$

Lemma 6.2.1. *Let φ_γ be as above. Then $\varphi_\gamma\left(X^*\left(G_{0,K}\right) \times P(R)\right) \subset \mathbb{Q}$.*

Proof. Let $\chi \in X^*\left(G_{0,K}\right)$ be a character. Then

$$\chi \circ \mathfrak{l}_0(1) = \mathrm{Lie}\left(\chi \circ p_0 \circ \gamma\right)(1)$$

is an integer, since $\chi \circ p_0 \circ \gamma$ is an endomorphism of the multiplicative group.

For a root system (V, S) let $Q(S)$ be the lattice generated by S in V. Then, for $i = 1, \cdots, r$, we have inclusions $Q(R_i) \subset X^*(H_i) \subset P(R_i)$, where both $Q(R_i), P(R_i)$ are lattices in the same vector space (so that the first has finite index in the second). In particular, for every weight $\lambda_i \in P(R_i)$ there exists $n_i \in \mathbb{Z}$ such that $n_i\lambda_i \in X^*(H_i)$, i.e. $n_i\lambda_i$ equals a certain character $\chi_i : H_i \to \mathbb{G}_m$. But then $n_i\lambda_i \circ \mathfrak{l}_i(1) = \mathrm{Lie}\,(\chi_i \circ p_i \circ \gamma)\,(1) \in \mathbb{Z}$ as before, so $\varphi_\gamma(\lambda_i) \in \mathbb{Q}$, as we wanted to show. □

Let us now suppose that $\rho : G \hookrightarrow GL_V$ is a finite-dimensional, faithful representation of G. Extending scalars to K we get a representation V_K of G_K; let W be an irreducible subrepresentation of V_K. The algebra $\mathfrak{g}$ being semi-simple by hypothesis, we have (thanks to Theorem 2.3.4) a decomposition

$$W \cong (K, \chi_0) \boxtimes (W_1, \rho_1) \boxtimes \cdots \boxtimes (W_n, \rho_n)$$

of W as external tensor product of irreducible representations of the factors $\mathfrak{c}$ and $\mathfrak{g}_i$. As G_0 is a torus, its only irreducible representations (over an algebraically closed field) are one-dimensional spaces where the action is given by a character χ_0.

Let now λ_i denote the highest weight of the representation W_i and X_i its full set of weights.

Lemma 6.2.2. *Consider the representation $\rho \circ \gamma : \mathbb{G}_{m,K} \to GL_{W,K}$ and let $N+1$ be the number of its different weights. Let $I = \{i | p_i \circ \gamma$ is not trivial$\}$.*

Then $\sum_{i \in I} l(\lambda_i) \leq N$.

Before proving this lemma we record a simple result we will need:

Lemma 6.2.3. *Let $\{A_i\}_{i=1,\cdots,r}$ be a finite family of finite subsets of $\mathbb{Q}$. Denote by $\displaystyle\sum_{i=1}^{r} A_i$ the subset of $\mathbb{Q}$ given by*

$$\{a_1 + \cdots + a_r | a_i \in A_i \quad \forall i = 1, \ldots, r\}.$$

Then $|\sum_{i=1}^{r} A_i| \geq \sum_{i=1}^{r} |A_i| - (r-1)$.

Proof. Induction shows that it is enough to do the case $r = 2$. For this, note that adding a to all the elements of A_2 and subtracting a from all the elements of A_1 leaves $A_1 + A_2$ untouched, so we can assume that all the elements in A_2 are non-negative and that 0 is one of them. Let $a_1^{(1)} < \cdots < a_{|A_1|}^{(1)}$ (resp. $a_1^{(2)} = 0 < \cdots < a_{|A_2|}^{(2)}$) be an enumeration of the elements of A_1 (resp. A_2). Then the numbers $a_1^{(1)} + 0, \cdots, a_{|A_1|}^{(1)} + 0, a_{|A_1|}^{(1)} + a_2^{(2)}, \cdots, a_{|A_1|}^{(1)} + a_{|A_2|}^{(2)}$ are all distinct by construction and belong to $A_1 + A_2$, so this last set has cardinality at least $|A_1| + |A_2| - 1$. □

We can now prove Lemma 6.2.2:

Proof. The set $P(W)$ of weights of ρ is clearly $\{\chi_0\} \times X_1 \times \cdots \times X_n$; the set of weights of $\rho \circ \gamma$ is then given by $\varphi_\gamma(P(W))$, so by definition of N we have

$$\begin{aligned} N+1 = |\varphi(P(W))| &= |\varphi_\gamma(\{\chi_0\} \times X_1 \times \cdots X_n)| \\ &= \left|\{\chi_0 \circ \mathfrak{l}_0(1)\} + \sum_{i=1}^n \varphi_\gamma(X_i)\right| = \left|\sum_{i \in I} \varphi_\gamma(X_i)\right|, \end{aligned}$$

where the last equality holds since trivial factors only contribute one weight. The above Lemma then yields the inequality

$$N+1 = \left|\sum_{i \in I} \varphi_\gamma(X_i)\right| \geq \sum_{i \in I} |\varphi_\gamma(X_i)| - (|I| - 1),$$

whence $N \geq \sum_{i \in I} (|\varphi_\gamma(X_i)| - 1)$; on the other hand, Lemma 2.4.13 implies that $|\varphi_\gamma(X_i)| - 1 \geq l(\lambda_i)$, so the claim follows. □

We now apply all of the above to the case of Mumford-Tate (and Hodge) groups. To this end, we take $k = \mathbb{Q}$ and $K = \mathbb{C}$, and for G the Mumford-Tate group M of a polarizable Hodge structure V (of pure weight n), which we know to be reductive. The representation $\rho : M \to GL_V$ will be the tautological one, and as cocharacter $\gamma : \mathbb{G}_{m,\mathbb{C}} \to M_\mathbb{C}$ we take $h_\mathbb{C} \circ \mu$. The action of z through $h_\mathbb{C} \circ \mu$ is given by

$$\bigoplus_{p \in \mathbb{Z}} z^p \operatorname{Id}_{V^{p,n-p}},$$

so each character of the representation γ corresponds to a non-trivial space $V^{p,n-p}$ in the Hodge decomposition of $V_\mathbb{C}$. We then have the following important theorem:

Theorem 6.2.4. *Keeping the above notation, let moreover $N+1$ be the number of integers p such that $V^{p,n-p} \neq (0)$. Then $l(\lambda_j) \leq N$ for every $j = 1, \ldots, r$.*

Proof. This follows immediately from Lemma 6.2.2 for those indices j such that γ has a non-trivial component in the simple factor G_j.

We can reduce to this case for every simple factor of M by exploiting the fact that this group is defined over $\mathbb{Q}$.

Fix a maximal torus T of M. The image of $\gamma : \mathbb{G}_{m,\mathbb{C}} \to M_\mathbb{C}$ is contained in a maximal torus $\tilde{T}$ of $M_\mathbb{C}$, and since all maximal tori are conjugated we find that $\tilde{T}$ is $M(\mathbb{C})$-conjugated to $T_\mathbb{C}$. But T is defined over $\mathbb{Q}$, hence it splits over $\overline{\mathbb{Q}}$, so the above conjugation argument yields another cocharacter (defined over $\overline{\mathbb{Q}}$) $\delta : \mathbb{G}_{m,\overline{\mathbb{Q}}} \to M_{\overline{\mathbb{Q}}}$ such that $\delta_\mathbb{C}$ is $M(\mathbb{C})$-conjugated to γ.

We can now make use of the action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ on this δ to get a cocharacter inducing non-trivial representations on each simple factor of $M_{\mathbb{C}}$. More precisely, for each $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ consider the cocharacter ${}^{\sigma}\delta$ and let $I_\sigma = \{i | {}^{\sigma}\delta \text{ has a non-trivial component in } G_i\}$. Let furthermore $I := \bigcup_{\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)} I_\sigma$. Then the algebraic group

$$M'_{\overline{\mathbb{Q}}} := Z(G)_{\overline{\mathbb{Q}}} \cdot \prod_{i \in I} G_{i,\overline{\mathbb{Q}}}$$

is normal in M (since it is an almost-direct product of normal subgroups) and is defined over $\mathbb{Q}$, since by definition of I the action of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ can only permute the factors appearing in the product. Now $M'_{\mathbb{C}}$ factors $\delta_{\mathbb{C}}$, $\delta_{\mathbb{C}}$ and γ are $M(\mathbb{C})$-conjugated, and M' is normal in M, hence $M'_{\mathbb{C}}$ factors γ. As $M' < M$, the minimality of the Mumford-Tate group (with respect to the property of factoring γ) implies $M' = M$, so every simple factor G_i of $M_{\mathbb{C}}$ appears in the product defining M': by construction of I, then, for every i there is a σ_i such that ${}^{\sigma_i}\delta$ has a non-trivial component in G_i. Since the representations afforded by ${}^{\sigma}\delta_{\mathbb{C}}, \delta_{\mathbb{C}}$ and γ clearly have the same number of non-trivial weights, the claim follows from Lemma 6.2.2 applied to ${}^{\sigma}\delta$ instead of γ. □

Corollary 6.2.5. *Let A be an Abelian variety, $V = H_1(A, \mathbb{Q})$ and M the Mumford-Tate group of A. Furthermore, let ρ_i denote the representation of the simple factor G_i of $M_{\mathbb{C}}$ induced by the tautological representation of M. Then, for every i, G_i is of classical type and every irreducible subrepresentation W_i of ρ_i is defined by a minuscule weight.*

Proof. Let ω_i denote the highest weight of W_i.

V has weight -1, hence in the Hodge decomposition of $V_{\mathbb{C}}$ we only have two non-trivial factors. With the notation of the above Theorem, this forces $N = 1$, so $l(\omega_i) = 1$, hence W_i is minuscule and G_i is of classical type because of Proposition 2.4.12. □

With rather different techniques, Pink has shown the ℓ-adic analogue of the above result, namely, he proves the following theorem (Corollary 5.11 of [Pin98]):

Theorem 6.2.6. *Each simple factor of the root system of G^0_ℓ has type A, B, C, or D, and its highest weights in the tautological representation are minuscule.*

6.3 A theorem of Pink

The paper [Pin98] also contains the following result, giving sufficient conditions (on the dimension and on the endomorphism algebra) for the Mumford-Tate conjecture to hold.

Theorem 6.3.1. *Suppose A is an Abelian variety of dimension g (defined over a number field K), such that*

- $\operatorname{End}(A) \cong \mathbb{Z}$
- *$2g$ is not a k^{th} power for any odd $k > 1$, nor of the form $\binom{2k}{k}$ for any odd $k > 1$.*

Then $MT(A) \cong CSp_{2g,\mathbb{Q}}$ and $G_\ell = CSp_{2g,\mathbb{Q}_\ell}$. In particular, the Mumford-Tate conjecture holds for A.

Proof. We take the uniform notation introduced in 6.1. Note that it is enough to prove that $H \cong Sp_{2g,\mathbf{Q}}$.

As a first step observe that A is simple, since if A is isogenous to $A_1^{e_1} \cdot \ldots \cdot A_n^{e_n}$, where the A_i's are simple and pairwise non-isogenous Abelian varieties, then

$$\operatorname{End}(A) \cong \prod_{i=1}^{n} M_{e_i \times e_i}\left(\operatorname{End}(A_i)\right),$$

where $M_{e\times e}(R)$ is the set of square matrices of order e with coefficients in R. The equality $\operatorname{End}(A) = \mathbb{Z}$ then clearly forces $n = 1$ and $e_1 = 1$; we also see that A is of type I in the Albert classification, so H is semisimple.

The hypothesis $\operatorname{End}(A) = \mathbb{Z}$ also implies that $V_{\mathbf{C}}$ is an irreducible representation of H:

$$\operatorname{End}(V \otimes \mathbf{C})^{\mathfrak{h}_{\mathbf{C}}} \cong \operatorname{End}(V)^{\mathfrak{h}} \otimes \mathbf{C} \cong \mathbb{Z} \otimes \mathbf{C} \cong \mathbf{C},$$

so the claim follows by Schur's lemma.

Write $H_{\mathbf{C}} \cong G_1 \cdot \ldots \cdot G_n$ for the decomposition of $H_{\mathbf{C}}$ as almost-direct product of its simple factors, $\mathfrak{g}_i$ for the Lie algebra of G_i and

$$V_{\mathbf{C}} \cong V_1 \boxtimes \cdots \boxtimes V_n$$

for the decomposition of $V_{\mathbf{C}}$ as (exterior) tensor product of irreducible representations of the factors $\mathfrak{g}_i$.

Lemma 6.3.2. *All the simple factors G_i are isomorphic to each other over $\overline{\mathbf{Q}}$. Moreover, all the representations V_i are isomorphic to each other.*

Proof. This is basically the same argument as in the proof of Theorem 6.2.4.

Consider the cocharacter $\delta : \mathbb{G}_{m,\overline{\mathbf{Q}}} \to M_{\overline{\mathbf{Q}}}$ introduced there. We show that δ has non-trivial components in exactly one simple factor.

Suppose this is not the case. If G_{j_1}, G_{j_2} are two factors in which δ has non-trivial components, let X_1, X_2 be the set of weights of $\mathfrak{g}_{j_1}, \mathfrak{g}_{j_2}$ that intervene in the Lie algebra representations induced by $h \circ p_{j_1} \circ \delta$ and $h \circ p_{j_2} \circ \delta$ (note that these applications are not well-defined as group morphisms, but do exist at the level of Lie algebras, since p_j induces an isomorphism between the Lie algebra of G_j and that of G'_j). By hypothesis $|X_1| \geq 2, |X_2| \geq 2$, so the representation induced on the Lie algebra of $H_{\mathbf{C}}$ by $h \circ \delta$ has at least $|X_1| + |X_2| - (2-1) \geq 3$ weights because of Lemma 6.2.3. But $h_{\mathbf{C}} \circ \delta_{\mathbf{C}}$ has as many weights as $h_{\mathbf{C}} \circ \mu$, and this last representation only has two, corresponding to the two non-trivial spaces in the Hodge (resp. Hodge-Tate) decomposition, contradiction.

It follows that δ only has non-trivial components in one simple factor, and the proof of Theorem 6.2.4 shows that the orbit of this simple factor under the action of $\operatorname{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$ must be the whole set of simple factors of H: these simple factors are therefore conjugated under the action of $\operatorname{Gal}\left(\overline{\mathbf{Q}}/\mathbf{Q}\right)$, hence all isomorphic.

The second statement then follows immediately from the transitivity of the Galois action. □

As H is a subgroup of the symplectic group by Lemma 3.2.9 (or its ℓ-adic counterpart), $V_{\mathbf{C}}$ is a symplectic representation of $H_{\mathbf{C}}$: Corollary 2.3.6 then implies that each factor V_i is self-dual. Moreover, all the algebras $\mathfrak{g}_i$ are isomorphic to each other, and the representations of the $\mathfrak{g}_i$'s afforded by the V_i's are isomorphic because of the above Lemma. Note furthermore that each (self-dual) factor is either orthogonal or symplectic, and the number of symplectic factors is odd (if it were even, the product would be orthogonal), so there is at least one symplectic factor. Since all the factors are isomorphic to each other by the above discussion, their number n is odd, for otherwise the product would again be orthogonal.

We then have

$$2g = \dim_{\mathbf{C}} V_{\mathbf{C}} = (\dim_{\mathbf{C}} V_1)^n$$

where n is odd, hence $n = 1$ by hypothesis. Using the irreducibility of V_1, it follows from Corollary 6.2.5 (resp. Theorem 6.2.6) that this representation is minuscule and the algebra $\mathfrak{g}_1$ is of classical type, so we just need to check the list given in theorem 2.4.6 for symplectic representations of classical algebras. We see that the following are the only possibilities:

- $\mathfrak{g}_1$ is of type A_{2s-1}, s odd, and $\dim V_1 = \binom{2s}{s}$;
- $\mathfrak{g}_1$ is of type B_l with $l \equiv 1 \text{ or } 2 \pmod 4$, and $\dim V_1 = 2^l$;

- $\mathfrak{g}_1$ is of type C_l, and $\dim V_1 = 2l$;
- $\mathfrak{g}_1$ is of type D_l, $l \equiv 2 \pmod 4$, and $\dim V_1 = 2^{l-1}$.

As $2g = \dim_{\mathbf{C}} V_{\mathbf{C}} = \dim_{\mathbf{C}} V_1$ is not of the form $\binom{2s}{s}$ for any odd $s > 1$, the first case cannot happen unless $s = 1$, but then $A_1 \cong C_1$ and we are in case (3).

Case (4) cannot happen either, since then $l-1$ would be odd and $2g$ would be an odd power (of 2), which is again against the hypotheses; finally, case (2) is impossible, too, since either l is odd, and then $\dim V_1$ is an odd power, or $l \equiv 2 \pmod 4$, and then $2^l = 4^{l/2}$ is again an odd power, which is against the hypotheses.

We deduce that we always are in case (3), hence $H_{\mathbf{C}} \cong Sp_{2g,\mathbf{C}}$ and $H \cong Sp_{2g,\mathbf{Q}}$. □

6.4 On the Lefschetz group

We now try to unwind the definition of $L(A)$, in order to get a more explicit description of this group. The result that follows (essentially taken from [Mur84], Lemma 2.3) gives a fairly clear picture of what $L(A)$ looks like.

We start by fixing our notation. Let A be a simple Abelian variety over $\mathbb{C}$, D its endomorphism algebra. Let furthermore E denote a fixed subfield of D, taken to equal D for varieties of type I and chosen among the maximal CM-subfields of D in the remaining cases, and let E^+ be the maximal totally real subfield of E.

Write V_λ for $V \otimes_{E^+,\lambda} \mathbb{R}$, where λ ranges through the set $\Sigma(E^+)$ of embeddings $E^+ \hookrightarrow \mathbb{R}$.

For varieties not of type I, V_λ has a natural complex structure coming from the action of $E \otimes \mathbb{R} \cong \bigoplus_{\lambda \in \Sigma(E^+)} \mathbb{C}$ on $V \otimes \mathbb{R} \cong \bigoplus_{\lambda \in \Sigma(E^+)} V_\lambda$. We shall write $V_\lambda^{\mathbb{C}}$ when V_λ is regarded as a $\mathbb{C}$-vector space through this action (in particular, note that $V_\lambda^{\mathbb{C}}$ is *not* the same object as $V_\lambda \otimes \mathbb{C}$).

Finally, let F be the center of D and F^+ its maximal totally real subfield. Then:

Proposition 6.4.1.
- *$L(A)_{\mathbb{R}} \subset \prod_\lambda GL(V_\lambda)$;*
- *the $L(A)_{\mathbb{R}}$-modules V_{λ_1} and V_{λ_2} are isomorphic if and only if $\lambda_1|_{F^+} = \lambda_2|_{F^+}$;*
- *if A is of type I, the projection of $L(A)_{\mathbb{R}}$ to $GL(V_\lambda)$ is $Sp\left(V_\lambda^{\mathbb{C}}, \omega_\lambda\right)$ with respect to an appropriate skew-symmetric form ω_λ on $V_\lambda^{\mathbb{C}}$;*

- *if A is of type II, the projection of $L(A)_{\mathbb{R}}$ to $GL(V_\lambda)$ can be described as the intersection*
$$U\left(V_\lambda^{\mathbb{C}}, \omega_\lambda^1\right) \cap Sp\left(V_\lambda^{\mathbb{C}}, \omega_\lambda^2\right),$$
where ω_λ^1 (resp. ω_λ^2) is a Hermitian (resp. skew-symmetric) form on $V_\lambda^{\mathbb{C}}$.

Proof. The first point amounts to saying that every V_λ is $L(A)$-stable, and it follows from arguments analogous to those of Section 1.1.4, essentially because the actions of $L(A)$ and E commute.

By definition of $L(A)$ we have the inclusion $L(A) \supseteq Hg(A)$, so

$$D = \operatorname{End}(V)^{Hg(A)} \supseteq \operatorname{End}(V)^{L(A)}.$$

On the other hand $\operatorname{End}(V)^{L(A)} \supseteq D$, as $L(A)$ commutes with D: it follows that the endomorphisms of V as a $Hg(A)$-representation and as an $L(A)$-representation are the same. Extending scalars to $\mathbb{C}$ we get a decomposition $V_{\mathbb{C}} \cong \prod_{\sigma \in \Sigma(E)} V_\sigma$; as E is a maximal subfield of D, the group $\operatorname{End}_{Hg(A),E}(V)$ - which equals the centralizer of E in D - is just E. By tensoring with $\mathbb{C}$ we get
$$\operatorname{End}\left(V_{\mathbb{C}}\right)^{Hg(A)} \cong E \otimes \mathbb{C} = \mathbb{C}^{\Sigma(E)},$$
so each V_σ is a simple representation of $Hg(A)_{\mathbb{C}}$, hence of $L(A)_{\mathbb{C}}$.

On the other hand, we could instead consider the decomposition of $V_{\mathbb{C}}$ with respect to the action of F. Write it as $V_{\mathbb{C}} \cong \prod_{\tau \in \Sigma(F)} W_\tau$. Then clearly $W_\tau = \{v \in V_{\mathbb{C}} | f \cdot v = \tau(f)v \quad \forall f \in F\}$ is the direct sum of the V_σ's such that $\sigma|_F = \tau$, and if q denotes $[E : F]$ every embedding of F in $\mathbb{C}$ admits exactly q different extensions. Comparing endomorphism algebras we get (note that $q \in \{1, 2\}$)

$$M_q(\mathbb{C}) \cong D \otimes_{F,\tau} \mathbb{C} = \operatorname{End}\left(W_\tau\right)^{Hg(A)_{\mathbb{C}}} = \prod_{\sigma_1, \sigma_2 : \sigma_i|_F = \tau} \operatorname{Hom}\left(V_{\sigma_1}, V_{\sigma_2}\right)^{Hg(A)_{\mathbb{C}}}.$$

Each space $\operatorname{Hom}^{Hg(A)_{\mathbb{C}}}\left(V_{\sigma_1}, V_{\sigma_2}\right)$ has dimension at most one (by Schur's lemma, which applies because each V_σ is simple); the equality

$$q^2 = \sum_{\sigma_1, \sigma_2 : \sigma_i|_F = \tau} \dim_{\mathbb{C}}\left(\operatorname{Hom}\left(V_{\sigma_1}, V_{\sigma_2}\right)^{Hg(A)_{\mathbb{C}}}\right)$$

then shows that each one of them has dimension exactly one: the sum on the right involves q^2 terms, so each one must contribute with exactly 1. It follows that $V_{\sigma_1} \cong V_{\sigma_2}$ as $L(A)$-modules (or, equivalently, $Hg(A)$-modules) if and only if $\sigma_1|_F = \sigma_2|_F$. Finally, since $V_\lambda = V_\sigma \oplus V_{\overline{\sigma}}$, we see that two V_λ's are isomorphic exactly when the corresponding λ's agree on the set of fixed points for the action of complex conjugation on F, i.e. when they agree on F^+.

As for the last two points, observe that the arguments in Section 1.1.4 yield the existence of an $L(A)$-invariant form ψ_λ on each factor V_λ, hence the projection of $L(A)$ to $GL(V_\lambda)$ is given by the set of D_λ-linear automorphisms of V_λ that preserve ψ_λ, where $D_\lambda := D \otimes_{F,\lambda} \mathbb{R}$.

If A is of type I, then D_λ is simply $\mathbb{R}$, so the condition of being D_λ-linear is void and the projection of $L(A)$ to $GL(V_\lambda)$ is simply $Sp(V_\lambda, \psi_\lambda)$, as claimed.

If A is of type II, then D is split at every infinite place, so $D_\lambda \cong M_2(\mathbb{R})$ and we can choose two elements $\alpha, \beta \in D_\lambda$ such that $\alpha^2 = 1, \beta^2 = -1, \beta\alpha = -\alpha\beta$ (this is clear in the matrix algebra: take for example $\alpha = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \beta = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$). This essentially identifies D_λ with $\mathbb{R} \oplus \mathbb{R}\beta \oplus \mathbb{R}\alpha \oplus \mathbb{R}\beta\alpha \cong \mathbb{C} \oplus \mathbb{C}\alpha$, and β can be chosen so that the natural action of $\mathbb{C}$ coming from $E \otimes_{\mathbb{Q}} \mathbb{R}$ coincides with the action of the first factor $\mathbb{C}$ in this decomposition. From the details of the proof of the Albert classification we know that the positive involution on D extends to an involution ρ on D_λ in such a way that, under the identification $D_\lambda \cong M_2(\mathbb{R})$, ρ becomes the usual transposition of matrices. We therefore have $\alpha^\rho = \alpha, \beta^\rho = -\beta$. We can then extend this ρ to (not necessarily square) matrices with coefficients in D_λ by setting $(a_{ij})^\rho = \left(a_{ij}^\rho\right)^t$. Note that this ρ is simply complex conjugation on the first factor $\mathbb{C}$ appearing in the decomposition $D_\lambda \cong \mathbb{C} \oplus \mathbb{C}\alpha$, but it is the identity on the second: indeed, our assumptions on α, β imply $(\beta\alpha)^\rho = \alpha^\rho\beta^\rho = \alpha\,(-\beta) = \beta\alpha$.

From now on, we shall think of V_λ as a module over $\mathbb{C} \oplus \mathbb{C}\alpha$; if d is the dimension of V_λ over D_λ, choosing bases allows us to write a vector v as $v_1 + v_2\alpha$ with v_1, v_2 (row) vectors in $\mathbb{C}^d$, and linear transformations of V_λ as $M_1 + M_2\alpha$ with $M_1, M_2 \in M_d(\mathbb{C})$. Recall now that ψ_λ is D_λ-skew-Hermitian, i.e. $\psi_\lambda(w, v) = -\psi_\lambda(v, w)^\rho$. By bilinearity we can write our form ψ_λ as

$$\psi_\lambda\,(v_1 + v_2\alpha, w_1 + w_2\alpha) = (v_1 + v_2\alpha)(T_1 + T_2\alpha)(w_1 + w_2\alpha)^\rho,$$

and using $w_2\alpha = \alpha\overline{w_2}$ (which follows from $\alpha\beta = -\beta\alpha$) we rewrite the above as

$$\psi_\lambda\,(v_1 + v_2\alpha, w_1 + w_2\alpha) = (v_1, v_2)M_1(\overline{w_1}, \overline{w_2})^t + (v_1, v_2)M_2(w_1, w_2)^t\alpha,$$

where M_1, M_2 are the following $2d \times 2d$ matrices:

$$M_1 = \begin{pmatrix} T_1 & T_2 \\ \overline{T_2} & \overline{T_1} \end{pmatrix} \quad M_2 = \begin{pmatrix} T_2 & T_1 \\ \overline{T_1} & \overline{T_2} \end{pmatrix}$$

The condition $\psi_\lambda(w,v) = -\psi_\lambda(v,w)^\rho$ implies $T^\rho = -T$, so $T^\rho = \overline{T_1}^t + T_2^t\alpha$ and T_1 is skew-Hermitian while T_2 is skew-symmetric. It is immediate to check that the same holds for M_1, M_2. It follows that a $\mathbb{C}$-linear endomorphism A of V_λ preserves ψ_λ if and only if it leaves invariant the Hermitian form ω_λ^1 associated to $-iM_1$ and the skew-symmetric form ω_λ^2 associated to M_2. But it in fact, an endomorphism preserving the two is automatically D_λ-linear: write π_1 (resp. π_2) for the projection of D_λ on $\mathbb{C}$ (resp. $\mathbb{C}\alpha$).

By our choices of α, β we have, $\pi_1(\alpha(z_1 + z_2\alpha)) = \pi_1(\overline{z_1}\alpha + \overline{z_2}) = \overline{z_2} = \overline{\pi_2(z_1 + z_2\alpha)}$, and similarly $\pi_2(\alpha(z_1 + z_2\alpha)) = \overline{\pi_1(z_1 + z_2\alpha)}$. Observe that preserving ω_λ^1 (resp. ω_λ^2) amounts to preserving the skew-Hermitian (resp. skew-symmetric) form $\pi_1 \circ \psi_\lambda$ (resp. $\pi_2 \circ \psi_\lambda$). Then, if A is a $\mathbb{C}$-linear automorphism preserving the two, we have

$$\begin{aligned}
\pi_1\left(\psi_\lambda(\alpha Av, Aw)\right) &= \pi_1\left(\alpha\psi_\lambda(Av, Aw)\right) && (D_\lambda\text{-linearity})\\
&= \overline{\pi_2\left(\psi_\lambda(Av, Aw)\right)} && (\pi_1 \circ \alpha = \overline{\pi_2})\\
&= \overline{\pi_2\left(\psi_\lambda(v, w)\right)} && (A \text{ preserves } \pi_2 \circ \psi_\lambda)\\
&= \pi_1\left(\alpha\psi_\lambda(v, w)\right) && (\pi_1 \circ \alpha = \overline{\pi_2})\\
&= \pi_1\left(\psi_\lambda(\alpha v, w)\right) && (D_\lambda\text{-linearity})\\
&= \pi_1\left(\psi_\lambda(A\alpha v, Aw)\right) && (A \text{ preserves } \pi_1 \circ \psi_\lambda);
\end{aligned}$$

similar computations show

$$\pi_2\left(\psi_\lambda(\alpha Av, Aw)\right) = \pi_2\left(\psi_\lambda(A\alpha v, Aw)\right),$$

so $\psi_\lambda(\alpha Av, Aw) = \psi_\lambda(A\alpha v, Aw)$, and by non-degeneracy $\alpha A = A\alpha$, so A is D_λ-linear.

Putting everything together, we have shown that the projection of $L(A)_\mathbb{R}$ to $GL(V_\lambda)$ is exactly the intersection

$$U\left(V_\lambda^\mathbb{C}, \omega_\lambda^1\right) \cap Sp\left(V_\lambda^\mathbb{C}, \omega_\lambda^2\right).$$

□

Corollary 6.4.2. *Let A be simple of type II and $\mathfrak{l}$ the Lie algebra of $L(A)$.*

Then the action of $\mathfrak{l}_\mathbb{C}$ on $V_\mathbb{C}$ can be described as follows: $\mathfrak{l}_\mathbb{C}$ is isomorphic to the product

$$\prod_{\lambda \in \Sigma(E^+ = F)} \mathfrak{sp}_{d,\mathbb{C}},$$

where

$$d = \dim_\mathbb{C} V_\lambda = \frac{1}{2}\dim_\mathbb{R} V_\lambda = \frac{1}{2}\cdot\frac{1}{[E^+ : \mathbb{Q}]}\dim_\mathbb{Q}(V) = \frac{2\dim(A)}{2[F:\mathbb{Q}]} = \frac{\dim(A)}{[F:\mathbb{Q}]},$$

and the representation it defines on $V_\mathbb{C}$ is isomorphic to two copies of $\boxtimes_\lambda \mathrm{Std}_\lambda$, where Std_λ is the standard representation of the copy of $\mathfrak{sp}_{d,\mathbb{C}}$ indexed by λ.

Proof. First note that $\mathfrak{l}_{\mathbb{R}}$ equals the product of its projections on the factors $\mathfrak{gl}(V_\lambda)$. This follows from Lemma 3.3.1, with condition (iii) in part (b) holding because the $\mathfrak{l}_{\mathbb{R}}$-modules V_λ are pairwise non-isomorphic, thanks to the second point of the previous Proposition and the fact that $F = F^+ = E^+$ in this case.

We want to understand what happens to the involved groups (and to their intersection) extending scalars to $\mathbb{C}$. Write, as usual,

$$V_\lambda \otimes_{\mathbb{R}} \mathbb{C} \cong V \otimes_F \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong V_\sigma \oplus V_{\overline{\sigma}},$$

where $\sigma, \overline{\sigma}$ are the two extensions of λ to E. Let us work at the level of Lie algebras. Choosing appropriate bases, we can assume that ω_λ^1 is the standard Hermitian form, so the Lie algebra of $U\left(V_\lambda^{\mathbb{C}}, \omega_\lambda^1\right)$ is simply

$$\mathfrak{u}\left(V_\lambda^{\mathbb{C}}\right) = \left\{M \in \mathrm{Mat}\,(d, \mathbb{C})\,|M + M^H = 0\right\}.$$

Extending scalars to $\mathbb{C}$ we have an isomorphism

$$\begin{array}{ccc} \mathrm{Mat}\,(d,\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} & \to & \mathrm{Mat}\,(d,\mathbb{C}) \oplus \mathrm{Mat}\,(d,\mathbb{C}) \\ M \otimes z & \mapsto & (Mz, \overline{M}z), \end{array}$$

that restricted to $\mathfrak{u}$ becomes $M \otimes z \mapsto \left(Mz, -(Mz)^t\right)$, so that composing with the projection on the first factor $\mathrm{Mat}\,(d,\mathbb{C})$ gives an isomorphism $\mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Mat}\,(d,\mathbb{C})$ (more abstractly, we are simply saying that U is an $\mathbb{R}$-form of GL).

Let ω_σ (resp. $\omega_{\overline{\sigma}}$) denote the skew-symmetric form induced by ω_λ^2 on V_σ (resp. $V_{\overline{\sigma}}$) through the above isomorphism. We claim that the image of $\mathfrak{u} \cap \mathfrak{sp}\left(V_\lambda^{\mathbb{C}}, \omega_\lambda^2\right)$ in $\mathfrak{gl}(V_\sigma)$ is simply $\mathfrak{sp}\left(V_\lambda^{\mathbb{C}}, \omega_\sigma\right)$. Indeed, on the one hand an element in the intersection $\mathfrak{u} \cap \mathfrak{sp}\left(V_\lambda^{\mathbb{C}}, \omega_\lambda^2\right)$ has to preserve the form ω_λ^2 induced on V_σ by the above isomorphism; on the other hand, any element B in $\mathfrak{sp}\left(V_\lambda^{\mathbb{C}}, \omega_\sigma\right)$ must come from an element A in $\mathfrak{u} \otimes \mathbb{C}$, and we only need to show that this A preserves ω_λ^2.

In order to do this, it is enough to show that B preserves both ω_σ and $\omega_{\overline{\sigma}}$. Note that B acts via A on V_σ and via $-A^t$ on $V_{\overline{\sigma}}$; fixing bases, if ω_σ is represented by a skew-symmetric matrix J, then $\omega_{\overline{\sigma}}$ is represented by $\overline{J}$, and the implications

$$A^t J + JA = 0 \Rightarrow A^H \overline{J} + \overline{JA} = 0 \Rightarrow (-A^t)^t \overline{J} + \overline{J}\left(-A^t\right) = 0$$

show that $-A^t$ preserves $\omega_{\overline{\sigma}}$, so we are done.

To describe the action, simply note that V_σ is by definition the standard representation of $\mathfrak{sp}\left(V_\lambda^{\mathbb{C}}, \omega\right)$, and we also know that an element A in this algebra acts as $-A^t$ on $V_{\overline{\sigma}}$, so $V_{\overline{\sigma}}$ is the contragradient representation. As the standard representation is clearly self-dual, being symplectic by definition, the two representations are in fact isomorphic to each other (and to the standard one), as claimed. $\square$

Remark 6.4.3. When F is a real quadratic field over $\mathbb{Q}$ and D is a quaternion algebra over F, we see that $\mathfrak{l}_{\mathbb{C}}$ is isomorphic to the product of $2 = [F : \mathbb{Q}]$ copies of $\mathfrak{sp}_{2,\mathbb{C}} \cong \mathfrak{sl}_2$, since (with the above notation) $d = \frac{\dim(A)}{[F:\mathbb{Q}]} = 2$.

Also note that the structure result we have just shown implies that $L(A)$ is reductive.

6.5 Varieties of Type I

Let (A, λ) denote a polarized, simple Abelian variety over $\mathbb{C}$ (resp. a number field K). We give here a sufficient condition for the equality $Hg(A) = L(A)$ to hold for varieties of type I.

Theorem 6.5.1. *Suppose A is of type I and let E be its endomorphism algebra (a totally real number field in this case). Let d be the degree of E over $\mathbb{Q}$ and suppose further that $\dim(A) = dh$, where h is either 2 or an odd integer.*

Then the rank of H is $\dim(A)$; in particular, $Hg(A)$ equals $L(A)$.

Proof. A is of type I, so $\mathfrak{h}$ is semisimple: extend scalars to $\mathbf{C}$ and write $\mathfrak{h}_{\mathbf{C}} \cong \prod_{i=1}^{t} \mathfrak{h}_i$, where each factor $\mathfrak{h}_i$ is a simple Lie algebra. In the ℓ-adic case, we can assume without loss of generality that ℓ splits completely in E (see Theorem 5.3.5).

Let $\Sigma(E)$ be the set of embeddings $\sigma : E \hookrightarrow \mathbb{C}$.

Upon extension of scalars, the tautological representation of $\mathfrak{h}$ on V becomes $V_{\mathbf{C}} \cong \prod_{\sigma \in \Sigma(E)} V_\sigma$, where the V_σ's are simple, pairwise non-isomorphic $\mathfrak{h}$ modules: we have

$$\operatorname{End}_{\mathfrak{h}_{\mathbf{C}}}(V_{\mathbf{C}}) = \operatorname{End}(V)^{\mathfrak{h}} \otimes \mathbf{C} \cong E \otimes \mathbf{C},$$

and this last space equals $\mathbb{C}^{\Sigma(E)}$ in the geometric case and $E_\ell \otimes \mathbf{C} \cong (\mathbb{Q}_\ell)^{[E:\mathbb{Q}]} \otimes \mathbf{C} \cong \mathbf{C}^{[E:\mathbb{Q}]}$ in the ℓ-adic one, so the claim follows from Schur's lemma.

Notice that the V_σ's coincide with those introduced in Section 1.1.4, so each one of them is a symplectic representation of $\mathfrak{h}$. Write ψ_σ for the non-degenerate, alternating form induced by ψ on V_σ and $V_\sigma = \bigotimes_{j=1}^{t} W_{\sigma,j}$ for the decomposition of V_σ as exterior product of simple $\mathfrak{h}_j$-modules.

We now want to show that this decomposition only has one non-trivial factor and deduce its precise representation structure. A first important remark is that

$$\dim_{\mathbf{C}} V_\sigma = \frac{\dim_{\mathbf{C}} V_{\mathbf{C}}}{[E : \mathbb{Q}]} = \frac{2\dim(A)}{d} = 2h$$

is either 4 or twice an odd number.

Consider first the case h odd. For each σ, there is at least one index j such that $W_{\sigma,j}$ is non trivial; this representation is then minuscule by Corollary 6.2.5 (resp. Theorem 6.2.6) and, since V_σ is symplectic, it is either symplectic or orthogonal by Corollary 2.3.6. Thanks to Theorem 2.4.6 we know that any self-dual, irreducible and minuscule representation is of even dimension: this is clear for types B_l, C_l and D_l, and for A_l follows from the fact that

$$\binom{2r}{r} = \frac{(2r)!}{r!r!} = \frac{2r}{r}\frac{(2r-1)!}{r!(r-1)!} = 2\binom{2r-1}{r}$$

is even (at least for $r \geq 1$). Then, for each σ, in the decomposition $\bigotimes_{j=1}^t W_{\sigma,j}$ exactly one factor is non-trivial, for otherwise $2h = \dim(V_\sigma)$ would be divisible by 4.

If $h = 2$, then either we have exactly one non-trivial factor (of dimension four, so that it is isomorphic to the standard representation of $\mathfrak{sp}_4$), or exactly two, each one of dimension 2: this cannot happen, since they would then be isomorphic to the standard representation of $\mathfrak{sl}_2$, and their product would be orthogonal, while we know it to be symplectic. In any case, we deduce that for each σ exactly one factor $W_{\sigma,j}$ is non-trivial: this non-trivial $W_{\sigma,i}$ must then be symplectic, as V_σ is, and of dimension $2h$; with this additional condition, the table of minuscule weights shows that $\mathfrak{h}_i$ is the symplectic algebra $\mathfrak{sp}_{2h,\mathbf{C}}$ and $W_{\sigma,i}$ is simply its standard representation (again, here we have to exclude algebras of type A_l: this is done by noticing that the possible dimensions for irreducible, minuscule and symplectic representations of algebras of type A_l are of the form $\binom{4k+2}{2k+1}$, and these numbers are divisible by 4).

For each index i let $\Sigma(i) = \{\sigma \in \Sigma | \mathfrak{h}_i \text{ acts non-trivially on } V_\sigma\}$. On one hand, since the representation of $\mathfrak{h}_\mathbf{C}$ afforded by $V_\mathbf{C}$ is faithful, we necessarily have $|\Sigma(i)| \geq 1$.

On the other hand, suppose by contradiction that for an index i we had $|\Sigma(i)| > 1$. Let σ_1, σ_2 be two different elements of $\Sigma(i)$. Then $V_{\sigma_1} \cong V_{\sigma_2}$ as $\mathfrak{h}_i$-modules, hence $V_{\sigma_1} \cong V_{\sigma_2}$ as $\mathfrak{h}$-modules (since the other simple factors of $\mathfrak{h}$ act trivially on $V_{\sigma_1}, V_{\sigma_2}$), but this contradicts the previous remark that the V_σ's are pairwise non-isomorphic. It follows that $|\Sigma(i)| = 1$ for every i, hence $d = |\Sigma| = \sum_{i=1}^t |\Sigma(i)| = t$. Let σ_i be the unique element of $\Sigma(i)$ and let $\mathfrak{l}_i$ be the Lie subalgebra of $\mathfrak{gl}(V_{\sigma_i})$ given by the endomorphisms preserving $\psi|_{V_{\sigma_i}}$.

From the above we see that the action of $\mathfrak{h}_\mathbf{C} \cong \prod_{i=1}^t \mathfrak{h}_i$ on $\bigoplus_i V_{\sigma_i}$ can be described as follows: the simple factor $\mathfrak{h}_i$ of $\mathfrak{h}_\mathbf{C}$ projects isomorphically onto $\mathfrak{l}_i$, which in turn acts tautologically on V_{σ_i}. Since all the automorphisms of $\mathfrak{l}_i \cong \mathfrak{sp}_{2h}$ are inner, we are exactly in the situation of Lemma 3.3.1, so $\mathfrak{h}_\mathbf{C} \cong \bigoplus_{i=1}^d \mathfrak{sp}_{2h}$. The rank of $\mathfrak{h}$ is therefore d times the rank of $\mathfrak{sp}_{2h}$ and the claim follows.

Finally, from the explicit description of $L(A)$ (Prop. 6.9) we know that over $\mathbb{C}$ its Lie algebra becomes isomorphic to $\mathfrak{sp}_{2h}^d$, so the inclusion $Hg(A) \subseteq L(A)$ must be an equality. □

6.6 Varieties of Type II

We now extend the result of the previous section to varieties of type II; the argument essentially carries through, with just a few minor modifications.

Let, as before, A be a simple polarized Abelian variety over $\mathbb{C}$ (resp. a number field K) and D be its endomorphism algebra, whose center we denote E. Let $d = [E : \mathbb{Q}]$.

Theorem 6.6.1. *Suppose A is of type II and $\dim(A) = 2dh$, where h is either 2 or an odd number.*

Then the rank of H is dh and $Hg(A) = L(A)$.

Proof. Keeping all of the notation from the previous proof, we write $\prod_{i=1}^{t} \mathfrak{h}_i$ for the decomposition of $\mathfrak{h}_{\mathbf{C}}$ in simple factors and $\Sigma(E)$ for the set of embeddings $\sigma : E \hookrightarrow \mathbb{C}$.

Here again we can assume that ℓ splits completely in E; note that D_ℓ is then a product of central simple algebras D_σ, each one of degree four over the copy of $\mathbb{Q}_\ell$ indexed by σ; over $\mathbf{C}$, the algebras D_σ split and become isomorphic to the standard 2×2 matrix algebra.

The modules V_σ appearing in the decomposition $V_{\mathbf{C}} \cong \prod_{\sigma \in \Sigma(E)} V_\sigma$ are not simple anymore: in fact,

$$\begin{aligned} \operatorname{End}_{\mathfrak{h}_{\mathbf{C}}}(V_{\mathbf{C}}) &= \operatorname{End}(V \otimes \mathbf{C})^{\mathfrak{h}_{\mathbf{C}}} = \operatorname{End}(V)^{\mathfrak{h}} \otimes \mathbf{C} \\ &\cong D \otimes \mathbf{C} \cong \operatorname{Mat}(2, \mathbf{C})^{\Sigma(E)}, \end{aligned}$$

so Schur's lemma implies that each V_σ splits as $W_\sigma^{\oplus 2}$ for a certain simple module W_σ.

To fix notations, write $V_{\mathbf{C}} \cong \bigoplus_\sigma \left(W_\sigma^{(1)} \oplus W_\sigma^{(2)} \right)$ where each $W_\sigma^{(i)}$ is irreducible and $W_{\sigma_1}^{(i)} \cong W_{\sigma_2}^{(j)}$ if and only if $\sigma_1 = \sigma_2$. The V_σ's are again symplectic representations of $\mathfrak{h}$, and we are going to show that

- each $W_\sigma^{(i)}$ is a symplectic representation;
- the submodule $W := \bigoplus_\sigma \left(W_\sigma^{(1)} \oplus (0) \right) \subset \bigoplus_\sigma \left(W_\sigma^{(1)} \oplus W_\sigma^{(2)} \right)$ is faithful as a representation of $\mathfrak{h}$.

The first claim follows easily: let ψ_σ be the non-degenerate, alternating form induced by ψ on V_σ. Then the restriction of ψ_σ to $W_\sigma^{(1)}$ is either zero or non-degenerate (as $W_\sigma^{(1)}$ is simple); in the second case we are done, and in the first, the non-degeneracy of ψ_σ in V_σ identifies $W_\sigma^{(1)} \cong W_\sigma^{(2),\vee}$, so that $W_\sigma^{(1)} \cong W_\sigma^{(2)}$ is at least self-dual. Now, if it were to be orthogonal, then clearly both $W_\sigma^{(1)}$ and $W_\sigma^{(2)}$ would be orthogonal, and so would be their sum V_σ, which we know to be symplectic, contradiction. Therefore $W_\sigma^{(1)}$ is symplectic and we are done.

As for the second claim, simply observe that $V_{\mathbf{C}} \cong W \oplus W$, so W is faithful if and only if $V_{\mathbf{C}}$ is, and this is clearly the case.

Still along the lines of the previous proof, let $W_\sigma^{(1)} = \bigotimes_{j=1}^t X_{\sigma,j}$ be the decomposition of $W_\sigma^{(1)}$ as exterior product of simple $\mathfrak{h}_j$ modules. Note that $\dim_{\mathbf{C}}\left(W_\sigma^{(1)}\right) = \frac{1}{2}\dim_{\mathbf{C}} V_\sigma = \frac{1}{2}\frac{2\dim(A)}{[E:\mathbb{Q}]} = 2h$, so this is either 4 or twice an odd number.

All the rest of the argument then carries through in the exact same way, hence $\mathfrak{h}_{\mathbf{C}} \cong \bigoplus_{i=1}^d \mathfrak{sp}_{2h}$, the only difference being that the action on $V_{\mathbf{C}}$ is given by *two* copies of the representation we had in the case of varieties of type I.

To finish the proof we simply need to quote the result of Remark 6.4.3, which - along with the above computation - ensures that the Lie algebras of $Hg(A)$ and $L(A)$ become isomorphic upon extension of scalars to $\mathbb{C}$, whence the two groups coincide. $\square$

CHAPTER 7

On the Mumford-Tate conjecture

We set out to prove results on the Hodge, Tate and Mumford-Tate conjectures for particular classes of simple Abelian varieties that satisfy additional requirements on the dimension g. In case g is a prime number we get, following Ribet, a full proof of all the three conjectures.

We retain all of the notation introduced in 6.1

7.1 The action of CM fields

We start by describing important invariants associated to the action of CM fields on Abelian varieties and by fixing our notation. Suppose E is a CM field acting on a Abelian variety A defined over a field K (the complex numbers or a number field) in such a way that $1 \in E$ acts as the identity, and let $\Sigma(E)$ be the set of embeddings of E in $\mathbb{C}$. Let $\overline{K}$ be a fixed algebraic closure of K. Then the Lie algebra of $A_{\overline{K}}$ is both a $\overline{K}$-module (thanks to the $\overline{K}$-variety structure on $A_{\overline{K}}$) and an E-module, with the two structures being compatible when restricted to $\mathbb{Q}$, so $L := \operatorname{Lie}\left(A_{\overline{K}}\right)$ acquires the structure of an $E \otimes \overline{K}$-module.

Let m denote $\dfrac{2\dim(A)}{[E:\mathbb{Q}]}$.

Now Galois theory implies that

$$\begin{array}{ccc} E \otimes_{\mathbb{Q}} \overline{K} & \xrightarrow{\sim} & \overline{K}^{\Sigma(E)} \\ e \otimes z & \mapsto & (\sigma \mapsto z \cdot \sigma(e)) \end{array}$$

is an isomorphism; on the E-vector space L this isomorphism induces a $\overline{K}$-linear isomorphism

$$L \cong \bigoplus_{\sigma \in \Sigma(E)} L_\sigma,$$

where

$$L_\sigma = \{l \in L | e \cdot l = \sigma(e) l \quad \forall e \in E\}.$$

We set $n_\sigma := \dim_{\overline{K}} L_\sigma$, and observe that L is a free module over $E \otimes \overline{K}$ if and only if for every σ we have $n_\sigma = n_{\overline{\sigma}}$.

Definition 7.1.1. We say that a quadratic imaginary field k **acts on** A **with multiplicities** $\{a, b\}$ if the pair $\{n_\sigma, n_\tau\}$ just defined (for the only two embeddings σ, τ of k in $\mathbb{C}$) equals $\{a, b\}$.

We collect here a few particular cases of results of Shimura (cf. [Shi63] and also Chapter 9 of [BL04]) that we will need in the next sections:

Proposition 7.1.2. *The integers n_σ have the following properties:*

- *Suppose E is an imaginary quadratic field. Then, if A is simple and $\dim A \geq 3$, n_σ and n_τ are both non-zero.*
- *Suppose A is a simple fourfold of type IV. Then its endomorphism algebra is automatically a field; if, furthermore, A is of type IV(2,1), then, up to a choice of notation, $(n_{\sigma_1}, n_{\tau_1}) = (2, 0)$ and $(n_{\sigma_2}, n_{\tau_2}) = (1, 1)$.*
- *There exists no simple Abelian fourfold of type III(2).*

An useful tool which will simplify some of the proofs that will follow is the following inequality, due to Ribet ([Rib80], p.87):

Theorem 7.1.3. *Suppose A is a simple Abelian variety defined over $\mathbb{C}$ of CM type. Then*

$$\operatorname{rank}(\mathfrak{h}) \geq \log_2(2g).$$

7.2 A theorem of Serre

We now state and prove a theorem, due to Serre (cf. Proposition 4 of [Ser67]), that gives a sufficient condition in order for a reductive subgroup of GL to be all of it.

Theorem 7.2.1. *Let V be a finite-dimensional over $\mathbf{Q}$ and suppose given a decomposition $V_\mathbf{C} = V_\mathbf{C}(0) \oplus V_\mathbf{C}(1)$.*

Let G be a subgroup of $GL(W)$. Suppose that

1. *G is reductive and connected;*

2. *the centralizer of G inside $GL(E)$ is reduced to the homotheties;*

3. *for every $x \in \mathbb{G}_{m,\mathbf{C}}$, G contains the operator $\rho_x := \operatorname{id}_{V_{\mathbf{C}}(0)} \oplus x \operatorname{id}_{V_{\mathbf{C}}(1)}$.*

Suppose furthermore that the dimensions of $V_{\mathbf{C}}(0)$ and $V_{\mathbf{C}}(1)$ are relatively prime. Then $G = GL(W)$.

Remark 7.2.2. In the absolutely simple case, both the Mumford-Tate group and G_ℓ satisfy the hypotheses of this theorem, the first with respect to the Hodge decomposition and the second with respect to the Hodge-Tate one.

Proof. To simplify notation let $W = V_{\mathbf{C}}, W_0 = V_{\mathbf{C}}(0), W_1 = V_{\mathbf{C}}(1)$ and $n_0 = \dim_{\mathbf{C}} W_0$, $n_1 = \dim_{\mathbf{C}} W_1$.

As G is reductive we can write it as the almost-direct product $G = Z(G) \cdot S$, where S denotes the derived subgroup of G. $Z(G)$ is clearly contained in the centralizer of G inside GL, so $Z(G)$ is contained in the one-dimensional torus of homotheties. If, by contradiction, it were finite, then G would be semi-simple, hence contained in SL, which contradicts (3), since $\det(\rho_x) = x^{n_1}$ does not always equal one. This implies that $Z(G)$ equals the torus of homotheties in GL.

For every $x, y \in \mathbb{G}_{m,\mathbf{C}}$, the operator $\rho_{x,y} = x \operatorname{id}_{W_0} \oplus y \operatorname{id}_{W_1}$ belongs to G, since it can be written as $x \operatorname{id}_W \cdot (\operatorname{id}_{W_0} \oplus (y/x) \operatorname{id}_{W_1})$, and both factors belong to G. This gives us a map of algebraic groups

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbf{C}} \times \mathbb{G}_{m,\mathbf{C}} & \to & G \\ (x,y) & \mapsto & \rho_{x,y} \end{array}$$

whose image is a 2-dimensional torus which we call Θ.

Θ also contains the 1-dimensional torus

$$\begin{array}{ccc} \Psi : \mathbb{G}_{m,\mathbf{C}} & \to & G \\ x & \mapsto & \rho_{x^{n_1}, x^{-n_0}}, \end{array}$$

and it is easy to check that $\det \Psi(x) = x^{n_1 n_0} x^{-n_0 n_1} = 1$, so $\Psi(x) \in S$.

We now want to show that S is in fact simple: suppose, on the contrary, the existence of a nontrivial decomposition $S \cong \frac{S_1 \times S_2}{N}$, where both S_1 and S_2 are not finite and N is a finite subgroup of the center of $S_1 \times S_2$. Such a decomposition induces a structure of $(S_1 \times S_2)$-module on W, and we have a corresponding decomposition $W \cong W' \otimes W''$, where W' is module over S_1 and W'' over S_2. Let Ξ be the connected component of the identity of the inverse image of $\Psi(\mathbb{G}_{m,\mathbf{C}})$ inside $S_1 \times S_2$. We have three different actions of Ξ on W, given by the projections of Ξ on S_1, S_2 and G. Fix an isomorphism $\sigma : \Xi \to \mathbb{G}_m$. Let χ_1, χ_2, χ be the corresponding characters, thought of as Laurent polynomials in σ with non-negative integer coefficients. We have $\chi = \chi_1\chi_2$, and on the other hand the description of the action of Ξ/N on W implies $\chi = n_0\sigma^a + n_1\sigma^b$ for certain integers a, b. Also, note that $a \neq b$, for otherwise Ξ/N would act on W through homotheties, which it does not. It follows that χ_1 divides $n_0\sigma^a + n_1\sigma^b$ as a polynomial with natural coefficients, but this last polynomial is essentially irreducible: the only possible factorizations are those of the form $d\sigma^c \cdot \left(\frac{n_0}{d}\sigma^{a-c} + \frac{n_1}{d}\sigma^{b-c}\right)$ with d a natural number and c an integer. This follows at once from the fact that if g_1, g_2 are two polynomials with natural coefficients and neither is a monomial, then g_1g_2 has at least three non-zero coefficients. The hypothesis $(n_0, n_1) = 1$ then forces $d = 1$, so one of χ_1, χ_2 is of the form σ^c. Assume without loss of generality that it is χ_1. Then the image of S_1 inside S acts trivially on W, contradicting the fact that W is clearly a faithful S-module.

We have thus shown that S is simple.

Let now $h = n_0 + n_1$. Ψ induces a map from μ_h to $Z(G)$, since for every h-th root of unity μ the operator $\Psi(\mu) = \rho_{\mu^{n_1},\mu^{-n_0}} = \rho_{\mu^{n_1},\mu^{h-n_0}} = \rho_{\mu^{n_1},\mu^{n_1}}$ is a homothety. Moreover, the restriction of Ψ to μ_h is injective, since the hypothesis $(n_0, n_1) = 1$ implies that we can recover μ from μ^{n_1}, μ^{-n_0}, hence μ from $\Psi(\mu)$. This implies that μ_h is at the same time a subgroup of SL, of the homotheties and of G, so it is a subgroup of the center of S.

We are then left with proving the following: if S is a simple algebraic group such that h divides $|Z(S)|$, $\rho : S \to GL(W)$ is a nontrivial representation and $\dim(W) = h$, then ρ induces an isomorphism $S \to SL(W)$.

This can be done most easily using the classification of algebraic groups. Suppose first that S is of type A_n. Then $h|(n + 1)$, so

$$n + 1 \geq h \Rightarrow \dim(S) \geq \dim(SL(W)),$$

which forces ρ (whose image has dimension equal to the dimension of S) to be surjective, and hence an isomorphism, since $SL(W)$ is simply connected.

Suppose on the contrary that S is not of type A_n. Then the classification of algebraic groups shows that the center of S has at most 4 elements, whence $h \leq 4$.

The possibility $h = 1$ is excluded, since in this case $GL(W)$ is a torus (so there is *no* nontrivial representation of the simple group S in $GL(W)$).

If $h = 2$, then $\dim(S) = \dim(SL_2) = 3$, and every algebraic group of dimension 3 is isomorphic to SL_2 (since we are working over an algebraically closed field).

If $h = 3$, then $S \cong E_6$, which is of dimension 78, so it does not admit a map towards SL_3 (of dimension 8) with at most finite kernel.

Finally, if $h = 4$, then S is of type D_n with $n \geq 4$, but all of these groups have a dimension much bigger than that of SL_4, so the same argument as above applies and this case cannot happen, either. □

7.3 The case of prime dimension

With the results we have established so far it is now easy to prove the following theorem (which is a minor modification of Theorem 3 in [Rib83]):

Theorem 7.3.1. *Suppose A is a simple Abelian variety with* $\mathrm{End}^0(A) = E$, *an imaginary quadratic field. Let n_σ, n_τ be the multiplicities of the action of E on A. Then, if n_σ and n_τ are relatively prime, $Hg(A) = L(A)$ and the Mumford-Tate conjecture holds.*

Proof. We show, in fact, that we can compute both $Hg(A)$ and its ℓ-adic counterpart. We treat the geometric and ℓ-adic case at the same time. Our purpose is to show that $\mathfrak{g}_\ell = \mathbb{Q}_\ell \cdot \mathrm{id} \oplus \mathfrak{u}(V_\ell/E_\ell)$ (and, analogously, that $\mathfrak{g} := \mathrm{Lie}(MT(A))$ coincides with $\mathbb{Q} \cdot \mathrm{id} \oplus \mathfrak{u}(V/E)$). Thanks to Theorem 5.3.5 we know that the rank of $\mathfrak{g}_\ell$ is independent of ℓ, so we can take ℓ to a prime that splits completely in E in two places of good reduction for A.

From now on, we take the general notation described above.

The above assumption on ℓ in the ℓ-adic setting and the fact that E is totally imaginary in the geometric case imply $V \cong V_\sigma \oplus V_\tau$ where V_σ and V_τ are absolutely irreducible $\mathfrak{g}$-modules: as usual, we can compute

$$\mathrm{End}\,(V \otimes \mathbf{C})^{\mathfrak{g}} \cong \mathrm{End}\,(V)^{\mathfrak{g}} \otimes \mathbf{C} \cong E \otimes \mathbf{C} \cong \mathbf{C}^2,$$

so that Schur's lemma ensures that V_σ and V_τ are absolutely irreducible (and non-isomorphic). Combining this decomposition (with respect to the action of E) and the Hodge (resp. Hodge-Tate) decomposition of $V_{\mathbf{C}}$ we get a finer splitting

$$V_{\mathbf{C}} \cong (V_\sigma(0) \oplus V_\tau(0)) \oplus (V_\sigma(1) \oplus V_\tau(1)),$$

where

$$\dim V_\sigma(0) = n_\sigma, \dim V_\tau(0) = n_\tau$$

and hence $\dim V_\sigma(1) = n_\tau$.

Consider now the projection of $\mathfrak{g}$ to $\mathfrak{gl}\,(V_\sigma)$. The space V_ℓ admits a Hodge-Tate decomposition with $n_0 = n_\sigma, n_1 = n_\tau$, so Theorem 7.2.1 applies: $(n_0, n_1) = 1$ by hypothesis, V_σ is absolutely irreducible (so the centralizer of this projection is reduced to the homotheties) and all the involved groups are reductive and connected because of our assumptions and Falting's Theorem 5.3.3.

The above proves

$$\mathrm{rank}\,(\mathfrak{g}^{ss}) \geq \mathrm{rank}\,(\mathfrak{gl}\,(V_\sigma)^{ss}) = \dim V_\sigma - 1 = n_0 + n_1 - 1.$$

Now since $\mathfrak{g} \subseteq \mathbf{Q} \cdot \mathrm{id} \oplus \mathfrak{u}\,(V/E)$ and

$$\mathrm{rank}\,(\mathfrak{u}\,(V/E)^{ss}) = \mathrm{rank}\,(\mathfrak{su}\,(V/E)) = \frac{1}{2}\dim_{\mathbf{Q}} V - 1 = n_0 + n_1 - 1,$$

we only need to show that rank $\left(\mathfrak{g}^{ab}\right) \geq 2$.

In order to do this, simply consider the cocharacter

$$\begin{array}{rcl} \mu : \mathbb{G}_{m,\mathbf{C}} & \to & GL\left(V_{\mathbf{C}}\right) \\ x & \mapsto & \begin{pmatrix} \mathrm{id} & \\ & x \end{pmatrix}, \end{array}$$

which we know to factor through the (geometric, resp. ℓ-adic) Mumford-Tate group, and compose with $\det_E : G_{\mathbf{C}} \to T_{E,\mathbf{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$. Then

$$\begin{aligned} \det(\mu(x)) &= \left(\det\left(\mu(x)|_{V_{\mathbf{C},\sigma}}\right), \det\left(\mu(x)|_{V_{\mathbf{C},\tau}}\right)\right) \\ &= \left(x^{\dim V_{\mathbf{C},\sigma}(1)}, x^{\dim V_{\mathbf{C},\tau}(1)}\right) \\ &= \left(x^{n_\tau}, x^{n_\sigma}\right) \end{aligned}$$

is not contained in the diagonal of $\mathbb{G}_m \times \mathbb{G}_m$; now clearly the image of the determinant map contains the diagonal (as the homotheties belong to MT, resp. G_ℓ), so the image of $\det_E$ contains at least a torus of rank 2. It follows that the rank of the Abelian part of $\mathfrak{g}$ is at least two, and we are done. □

Combining all we know we finally get the following Theorem:

Theorem 7.3.2. *Let A be a simple Abelian variety of prime dimension p, defined over a number field K. Then the Hodge, Tate and Mumford-Tate conjectures hold for A.*

Remark 7.3.3. The theorem in itself is nothing new and it is essentially due to Ribet ([Rib83], Theorems 1,2 and 3), although he is only interested in the Hodge setting and makes no mention of the Mumford-Tate conjecture.

The validity of the Mumford-Tate conjecture for A of type I is a particular case of Theorem C in [BGK03], which in turn follows at once from the results of Section 6.5; furthermore, in case the endomorphism algebra is simply $\mathbb{Z}$, this was already known to Serre ([Ser86]).

Finally, as already remarked, the Mumford-Tate conjecture is known in full generality for varieties of CM type.

Proof. Because of the Albert classification we only have four possibilities for $\operatorname{End}^0(A)$, namely,

- Type I(1), $D = \mathbb{Q}$: in this case, as it is easy to check, Pink's Theorem applies, so $MT(A) \cong CSp_{2g,\mathbb{Q}}$ and $G_\ell(A) \cong CSp_{2g,\mathbb{Q}_\ell}$. Theorem 5.1.13 then concludes the proof for the Hodge conjecture and its ℓ-adic analogue implies the Tate conjecture for both A and all of its powers.

- Type I(p), $D = E$ is a totally real field of degree p over $\mathbb{Q}$: in this case the result of Section 6.5 applies (in a particularly simple form), so $\mathrm{Hg}(A) = L(A)$ and we conclude as before.

- Type IV(1,1), $D = k$ is a totally imaginary quadratic field over $\mathbb{Q}$: the claim follows immediately from the above Theorem 7.3.1.

- Type IV(p,1), $D = E$ is a CM field of degree $2p$ over $\mathbb{Q}$: we know that the Mumford-Tate conjecture holds for CM varieties, so it is enough to work out the geometric case. Being in the CM case, we are dealing with algebraic tori by Proposition 3.1.10, so we only need to compare ranks. Introduce, as usual, algebraic tori

  $$T_E := \mathrm{Res}_{E/\mathbb{Q}}\,\mathbb{G}_m,\ T_{E_0} := \mathrm{Res}_{E_0/\mathbb{Q}}\,\mathbb{G}_m,$$

  and the normal closure L of any image of E in $\mathbb{C}$.

  We know that the Hodge group H is a subgroup (subtorus) of the kernel U_E of the norm map $N_{E/E_0} : T_E \to T_{E_0}$, so its rank is at most $\frac{1}{2}\dim T_E = p$, and what we need to show is $\mathrm{rank}(H) \geq p$. We distinguish two cases, according to whether the dimension of A is 2 or an odd prime.

  If the dimension is 2, we checked in Section 4.2 that $Hg(A) = L(A)$, and once again we conclude by Theorem 5.1.13.

  Note that another possible approach is to use Ribet's inequality (Theorem 7.1.3), that immediately yields the desired result:

  $$\mathrm{rk}(\mathfrak{h}) \geq \log_2(2\dim(A)) = 2.$$

  If p is odd, consider the character group $X^*(H)$, which is a quotient of $X^*(U_E)$. Identifying $V \cong E$ as E-modules, it is clear that the characters of T_E appearing in the representation V are simply the embeddings $\sigma : E \hookrightarrow L$. The H-module V is still multiplicity-free (since $\mathrm{End}(V_{\mathbb{C}}) = E \otimes \mathbb{C}$ is a direct sum of copies of $\mathbb{C}$), so the images of the various σ's in the quotient $X^*(H)$ are all distinct.

  Let now $\rho : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{Aut}\left(X^*(H)\right)$ be the morphism describing the action of the absolute Galois group of $\mathbb{Q}$ on the set of embeddings, K be its kernel and C its image. Clearly, if an element of $\mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ fixes L, it also fixes $X^*(H)$, and by the above remark the converse is also true, hence $K = \mathrm{Gal}\left(\overline{\mathbb{Q}}/L\right)$ and $C = \mathrm{Gal}\left(L/\mathbb{Q}\right)$. It follows that $|C| = [L : \mathbb{Q}]$ is divisible by $[E_0 : \mathbb{Q}] = p$, so C contains an element g of exact order p.

  The action of this g on $Y = X^*(H) \otimes \mathbb{Q}$ makes the latter into a module over $\mathbb{Q}[x]/(x^p - 1) \cong \mathbb{Q}(\zeta_p) \oplus \mathbb{Q}$, so we get a corresponding decomposition $Y = Y_1 \oplus Y_2$, where Y_1 is a $\mathbb{Q}(\zeta_p)$-vector space and Y_2 is a $\mathbb{Q}$-vector space.

Since we already know that g acts nontrivially on Y, Y_1 is nonzero, and hence its dimension over $\mathbb{Q}$ is at least $\dim_{\mathbb{Q}} \mathbb{Q}(\zeta_p) = p-1$. In order to show that $\operatorname{rank}(X^*(H)) \geq p$ we only need to show $\dim_{\mathbb{Q}}(Y) \geq p$, so it suffices to prove $Y_2 \neq 0$. In turn, this boils down to describing an element invariant under the action of g: take any embedding $\sigma : E \hookrightarrow \mathbb{C}$ and consider

$$\tilde{\sigma} := \sigma + g \cdot \sigma + \ldots + g^{p-1} \cdot \sigma,$$

which is clearly an invariant for the action of g. All we need to show now is $\tilde{\sigma} \neq 0$.

Consider the natural pairing $X^*(H) \times X_*(H) \to \mathbb{Z}$.

$X_*(H)$ is a $\operatorname{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$-submodule of $X_*(T_E)$, which is again the free Abelian group on the embeddings $\sigma : E \to \mathbb{C}$ (although with a different structure). Let Σ be the set of embeddings $E \hookrightarrow \mathbb{C}$ that appear as characters in the representation $H^{(1,0)}(A, \mathbb{C})$ of T_E: then $\sum_{\sigma \in \Sigma} \sigma$ is an element of $X_*(MT(A))$.

To see this, simply consider the usual cocharacter $\mu : \mathbb{G}_{m,\mathbb{C}} \to MT(A)_{\mathbb{C}}$ that defines the Mumford-Tate group, and observe that it is identified to the sum of those embedding appearing in $H^{(1,0)}(A, \mathbb{C})$. In particular, for every pair $(\sigma, \bar{\sigma})$, exactly one appears in this representation (since the other acts nontrivially on $H^{(0,1)}(A, \mathbb{C})$, so the claim follows by counting dimensions). Moreover, since $X_*(H)$ is the subgroup of elements of $X_*(MT(A))$ the sum of whose coefficients is zero, we see that

$$\chi := \sum_{\sigma \in \Sigma} (\sigma - \overline{\sigma}) \in X_*(H),$$

using the fact that the cocharacter group $X_*(MT(A))$ is stable under complex conjugation. As the coefficient of each embedding σ in χ is ± 1, the natural pairing $\langle \chi, \tilde{\sigma} \rangle$ is a sum of p terms, each of which is ± 1, hence it is an odd integer. In particular, it is not zero, so $\tilde{\sigma} \neq 0$ and we are done.

$\square$

CHAPTER 8

Simple Abelian fourfolds

To further our investigation of the Hodge and Tate conjectures for Abelian varieties we now move to simple Abelian fourfolds and try to connect the existence of exceptional (Hodge or Tate) classes on A with certain arithmetic properties of its endomorphism algebra.

Note that the many theorems of Chapter 5 assure us that the Hodge and Tate classes in H^2 and H^6 are in the algebra generated by divisor classes, so from now on we restrict our attention to H^4.

We keep using the notation from Chapter 6, but in order to describe the criterion proved in [MZ95] we shall need one more definition:

Definition 8.0.4. Let A be an Abelian variety. A sub-algebra B of $End^0(A)$ is said to be **stable under all Rosati involutions** if for every choice of a polarization φ on A we have $i_\varphi(B) = B$, where i_φ is the Rosati involution induced by φ.

We are now ready to state and prove the main theorem of [MZ95].

Theorem 8.0.5. *Let A be a simple Abelian fourfold over $\mathbb{C}$ (resp. a number field K). Then A supports exceptional Hodge (resp. Tate) classes if and only if the following hold:*

- $\mathrm{End}^0(A)$ *(resp.* $\mathrm{End}^0(A_{\overline{K}})$*) contains an imaginary quadratic field k such that k is stable under all Rosati involutions;*

- $\mathrm{Lie}(A)$ *becomes a free $(k \otimes_{\mathbb{Q}} \mathbb{C})$-module (resp.* $\mathrm{Lie}(A_{\overline{K}})$ *becomes a free $\left(k \otimes_{\mathbb{Q}} \overline{K}\right)$-module).*

8.1 Exceptional classes: sufficient condition

The 'if' part the theorem actually holds in arbitrary dimension, so for the moment we drop the assumption $\dim(A) = 4$.

We recast the previous Theorem in a slightly different form:

Theorem 8.1.1. *Let A be a simple Abelian variety over $\mathbb{C}$ (resp. a number field K). Assume that either of the following holds:*

- *A is of type III;*
- $\mathrm{End}^0(A)$ *(resp.* $\mathrm{End}^0(A_{\overline{K}})$*) is a CM-field E containing a CM-subfield F such that $n_\sigma = n_{\sigma'}$ for every $\sigma \in \Sigma(F)$.*

Then A supports exceptional Hodge (resp. Tate) classes.

As a first step we are going to show that the above assumptions are in fact equivalent to the original assumptions in Theorem 8.0.5. We shall need a lemma:

Lemma 8.1.2. *Let A be a polarized Abelian variety over $\mathbb{C}$ (resp. a number field K) and suppose E is a CM subfield of* $\mathrm{End}^0(A)$ *(resp.* $\mathrm{End}^0\left(A_{\overline{K}}\right)$*). Let furthermore $V = H_1(A, \mathbb{Q})$ and $V_\ell = V \otimes \mathbb{Q}_\ell$.*

Then, with the above notation, $n_\sigma = n_{\overline{\sigma}}$ for every embedding $\sigma : E \hookrightarrow \mathbb{C}$ if and only if $\mathfrak{h} \subseteq \mathfrak{su}(V/E)$ (resp. $\mathfrak{h}_\ell \subseteq \mathfrak{su}(V_\ell/E_\ell)$).

Proof. We start with the complex case. Write

$$\mathrm{Lie}(A) \otimes_{\mathbb{R}} \mathbb{C} \cong V \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{\sigma \in \Sigma(E)} H_\sigma.$$

From the Hodge decomposition of V we get a decomposition of every space H_σ, given by $H_\sigma = H_\sigma \cap (V \otimes_{\mathbb{Q}} \mathbb{C}) = H_\sigma \cap \left(V^{(-1,0)} \oplus V^{(0,-1)}\right) =: H_\sigma^{(-1,0)} \oplus H_\sigma^{(0,-1)}$. Let $a_\sigma := \dim\left(H_\sigma^{(-1,0)}\right), b_\sigma = \dim\left(H_\sigma^{(0,-1)}\right)$. As complex conjugation exchanges $\sigma, \overline{\sigma}$ and $H_\sigma, H_{\overline{\sigma}}$, we see that $H_\sigma^{(-1,0)} \oplus H_{\overline{\sigma}}^{(0,-1)}$ (being stable under complex conjugation) comes from a real vector space that can be identified with the space L_σ introduced at the beginning of Chapter 7. In particular, $a_\sigma = n_\sigma, b_\sigma = n_{\bar{\sigma}}$, so we have the equivalences

$$\begin{gathered}\mathrm{Lie}(A) \text{ becomes a free module over} \\ E \otimes_{\mathbb{Q}} \mathbb{C} \Longleftrightarrow n_\sigma = n_{\overline{\sigma}} \quad \forall \sigma \in \Sigma(E) \Longleftrightarrow a_\sigma = b_\sigma \quad \forall \sigma \in \Sigma(E).\end{gathered}$$

On the other hand, we always have the inclusion $\mathfrak{h} \subseteq \mathfrak{u}(V/E)$, so we just need compute the trace of an endomorphism belonging to the Hodge algebra.

To this end, notice that (letting $m = \dim_E(V) = \frac{2\dim(A)}{[E:\mathbb{Q}]}$)

$$\left(\bigwedge_E^m V\right) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigwedge_{E\otimes_{\mathbb{Q}}\mathbb{C}}^m (V \otimes_{\mathbb{Q}} \mathbb{C}) \cong \bigwedge_{E\otimes_{\mathbb{Q}}\mathbb{C}}^m \bigoplus_{\sigma\in\Sigma(E)} \left(H_\sigma^{(-1,0)} \oplus H_\sigma^{(0,-1)}\right)$$
$$\cong \bigoplus_{\sigma\in\Sigma(E)} \bigwedge_E^m \left(H_\sigma^{(-1,0)} \oplus H_\sigma^{(0,-1)}\right).$$

Expanding the internal exterior power (and using $m = a_\sigma + b_\sigma$) we then get

$$\bigoplus_{\sigma\in\Sigma(E)} \left(H_\sigma^{(-1,0)}\right)^{\oplus a_\sigma} \otimes \left(H_\sigma^{(0,-1)}\right)^{\oplus b_\sigma},$$

which is a Hodge structure with weights $(a_\sigma, b_\sigma)_{\sigma\in\Sigma(E)}$. Now an element $h \in \mathfrak{h}$ acts on $\bigwedge_E^m V$ as the multiplication by its trace as an E-linear endomorphism of V, so its trace is zero if and only if its action is trivial, which in turn happens exactly when all the pairs (a_σ, b_σ) satisfy $a_\sigma = b_\sigma$. Combining this last remark with the above chain of equivalences we get the desired result.

If A is defined over a number field K, fix an embedding $K \hookrightarrow \mathbb{C}$ and identify $\Sigma(E) = \operatorname{Hom}(E, \overline{K})$ with $\operatorname{Hom}(E, \mathbb{C})$.

If E acts on A in such a way that $n_\sigma = n_{\overline{\sigma}}$, then the same is true for its action on $A_{\mathbb{C}}$; it follows (using Deligne's result 5.3.6) that for any prime ℓ

$$\mathfrak{h}(A_{\mathbb{C}}) \subseteq \mathfrak{su}(V/E) \Rightarrow \mathfrak{h}_\ell \subseteq \mathfrak{h}(A_{\mathbb{C}}) \otimes \mathbb{Q}_\ell = \mathfrak{su}(V_\ell/E_\ell).$$

Conversely, suppose $\mathfrak{h}_\ell \subseteq \mathfrak{su}_{E_\ell}(V_\ell, \psi_\ell)$. Fix a place w of K above ℓ and a place $\overline{w}$ of $\overline{K}$ above w, and use Theorem 5.2.1 to get a Hodge-Tate decomposition $V_{\mathbf{C}} \cong V(0) \oplus V(1)$. The assumption means that $\operatorname{Gal}\left(\overline{K}/K\right)$ acts on $\bigwedge_{E_\ell}^m V_\ell$ via the character $\chi^{m/2}$, so this last module is purely of type $\left(\frac{m}{2}, \frac{m}{2}\right)$.

Comparing with the action of E_ℓ on the Hodge-Tate decomposition

$$V_{\mathbf{C}} \cong \bigoplus_{\sigma\in\Sigma} V_{\mathbf{C},\sigma} \cong \bigoplus_{\sigma\in\Sigma} (V_{\mathbf{C},\sigma}(0) \oplus V_{\mathbf{C},\sigma}(1))$$

we get, as before, $n_\sigma = \dim V_{\mathbf{C},\sigma}(0)$, whence the same proof as above yields

$$\left(\bigwedge_{E_\ell}^m V_\ell\right) \otimes_{\mathbb{Q}_\ell} \mathbf{C} \cong \bigwedge_{E_\ell\otimes_{\mathbb{Q}_\ell}\mathbf{C}}^m V_{\mathbf{C}} \cong \bigoplus_{\sigma\in\Sigma} \left(\bigwedge_{\mathbf{C}}^m V_{\mathbf{C},\sigma}\right)$$
$$\cong \bigoplus_{\sigma\in\Sigma} \left(\bigwedge^{n_\sigma} V_{\mathbf{C},\sigma}(0) \otimes \bigwedge^{m-n_\sigma} V_{\mathbf{C},\sigma}(1)\right),$$

which is purely of type $\left(\frac{m}{2}, \frac{m}{2}\right)$ if and only if $n_\sigma = n_{\sigma'} = m/2$ for every embedding σ. □

Remark 8.1.3. It is worth noting that we have also proved $a_\sigma + a_{\overline{\sigma}} = a_\sigma + b_\sigma = m = \frac{2\dim(A)}{[E:\mathbb{Q}]}$. We shall need this result again.

We can now resume the reduction of the 'if' part of Theorem 8.0.5 to Theorem 8.1.1. Observe that if $f \mapsto f^\dagger$ is one Rosati involution on $\mathrm{End}^0(A)$, then every other such involution is of the form $f \mapsto e^{-1} f^\dagger e$ for a certain e with $e = e^\dagger$ (thanks to the description of polarizations over an algebraically closed field).

Suppose F is a CM-field contained in $\mathrm{End}^0(A)$, and fix a polarization inducing complex conjugation on F. If a Rosati involution stabilizes F, then it acts as complex conjugation (since this is the only positive involution over a CM field); it follows that F is stable under all Rosati involutions if and only if for every e with $e = e^\dagger$ we have $e^{-1} f^\dagger e = f^\dagger$. As f ranges through F, so does $f^\dagger$, hence the previous condition is equivalent to $e^{-1} f e = f$ for every $\dagger$-symmetric e in E and for every f in F, so F has to commute with the algebra $\mathscr{S}$ generated by the $\dagger$-symmetric elements of $\mathrm{End}^0(A)$. Extending scalars to $\mathbb{R}$ it is easy to see that for varieties of type II this algebra coincides with the full endomorphism algebra, while for type III $\mathscr{S}$ is simply the center of $\mathrm{End}^0(A)$.

Depending on the type of A in the Albert classification we then have the following cases:

Type I The endomorphism algebra of A does not contain any imaginary quadratic field, so the conditions in the two formulations of the theorem are trivially equivalent.

Type II As remarked above, an element f of an imaginary quadratic field F stable under all Rosati involutions must lie in the center of $\mathrm{End}^0(A)$, which is real, so for varieties of Type II there never exists a quadratic imaginary field stable under *all* Rosati involutions.

Type III In this case, being stable for one Rosati involution is equivalent to being stable for any Rosati involution; if k is an imaginary quadratic field contained in $\mathrm{End}^0(A)$ (and clearly there always is such a field) we can choose a polarization that induces complex conjugation on k, so k is stable under at least one Rosati involution, hence under all Rosati involutions. Moreover, the above Lemma 8.1.2 says that the condition about $\mathrm{Lie}(A)$ becoming a free $(E \otimes \mathbb{C})$-module is always met for varieties of type III: indeed, it is equivalent to $\mathfrak{h} \subseteq \mathfrak{su}(V/E)$, but this follows from $\mathfrak{h} \subseteq \mathfrak{u}(V/E)$ by semisimplicity of $\mathfrak{h}$.

Type IV In this case, as $\mathrm{End}^0(A)$ is commutative, stability with respect to one Rosati involution is equivalent to stability under all Rosati involutions.

The above analysis reduces the 'if' part in the main result to Theorem 8.1.1. Before attacking the proof of the theorem itself we establish one more lemma:

Lemma 8.1.4. *Let E be a number field and V be a finite-dimensional E-vector space. Then, for every m, $\bigwedge_E^m V$ is, in a natural way, a direct summand of $\bigwedge_{\mathbb{Q}}^m V$.*

Proof. By the universal property of $\bigwedge_{\mathbb{Q}}^m V$ we have a natural map

$$\bigwedge_{\mathbb{Q}}^m V \twoheadrightarrow \bigwedge_E^m V;$$

similarly, replacing V with V^*, we have a surjective map $\bigwedge_{\mathbb{Q}}^m (V^*) \twoheadrightarrow \bigwedge_E^m (V^*)$. As it is well-known, exterior powers commute with taking duals, so we can consider the second map as a surjection $\left(\bigwedge_{\mathbb{Q}}^m V\right)^* \twoheadrightarrow (\bigwedge_E^m V)^*$, and this clearly gives rise to an injection $\bigwedge_E^m V \hookrightarrow \bigwedge_{\mathbb{Q}}^m V$. Using bases it is not difficult to check that the two maps are one the inverse of the other. □

An example by Mumford (that can be found in [Poh68]), further clarified by Weil in [Wei80], brings us to consider a particular family of cohomology classes on an Abelian variety. We give the following

Definition 8.1.5. Let A be an Abelian variety of dimension g, defined over K (the complex numbers or a number field). Suppose K is large enough, so that every endomorphism of A is defined over K; furthermore, let E be a field, $\nu : E \hookrightarrow \mathrm{End}^0(A)$ be a ring injection and $m = \frac{2\dim(A)}{[E:\mathbb{Q}]}$.

The space of **E-Weil classes** of A is

$$\mathscr{W}_E(A) = \bigwedge_E^m H^1(A, \mathbb{Q})$$

in the complex case, and

$$\mathscr{W}_{\ell,E}(A) = \bigwedge_{E\otimes\mathbb{Q}_\ell}^m H^1_{ét}\left(A_{\overline{K}}, \mathbb{Q}_\ell\right)$$

in the ℓ-adic case.

In case A is a fourfold, we define

$$\mathscr{W}(A) = \sum_k \left(\bigwedge_k^4 H^1(A, \mathbb{Q})\right),$$

where the sum runs over all imaginary quadratic subfields of $\mathrm{End}^0(A)$ acting on A with multiplicities $(n_\sigma, n_\tau) = (2, 2)$.

Analogously, if A is a fourfold defined over a number field K, for a fixed prime ℓ we define

$$\mathscr{W}_\ell(A) = \sum_k \left(\bigwedge_{k\otimes\mathbb{Q}_\ell}^4 H^1_{\acute{e}t}\left(A_{\overline{K}}, \mathbb{Q}_\ell\right) \right).$$

We will often refer to $\mathscr{W}(A), \mathscr{W}_\ell(A)$ simply as 'the spaces of Weil classes on A'.

Remark 8.1.6. Note that the above definition is not completely standard, but will be very useful in our case.

Let A be a simple fourfold and k be an imaginary quadratic field as in the above definition: it follows from Lemma 8.1.2 that the Hodge Lie algebra (resp. $\mathfrak{h}_\ell$) acts trivially on $\mathscr{W}(A)$ (resp. $\mathscr{W}_\ell$), so these are in fact Hodge (resp. Tate) classes, and it is a meaningful question to ask whether or not they are exceptional.

The Weil classes will in fact turn out to be the only exceptional classes that can appear on simple Abelian fourfolds. We state this fact as a theorem, which we will verify alongside the 'only if' part of Theorem 8.0.5.

Theorem 8.1.7. *Let A be a simple Abelian variety of dimension 4 over $\mathbb{C}$ (resp. a number field K). Then $\mathscr{B}^2(A) = \mathscr{D}^2(A) + \mathscr{W}(A)$ (resp. $\mathscr{T}_\ell^2(A) = \mathscr{D}_\ell^2(A) + \mathscr{W}_\ell(A)$).*

We are now ready to prove the 'if' part of the main theorem:

Proof. (Theorem 8.1.1) We start by showing that the assumptions imply the existence of two CM-subfields $F_1 \subseteq F_2 \subseteq \mathrm{End}^0(A) =: D$ such that

- $\mathfrak{h} \subseteq \mathfrak{su}(V/F_1)$;
- $F_2 \supseteq \left\{e \in \mathrm{End}^0(A) | e^\dagger = e\right\}$.

We distinguish two cases:

- if A is of type III, then D is a quaternion algebra over a totally real field E. Choose a quadratic extension F/E that splits D. Then

$$F \otimes_\mathbb{Q} \mathbb{R} \subseteq D \otimes_\mathbb{Q} \mathbb{R} = \prod_{i=1}^{[E:\mathbb{Q}]} \mathbb{H},$$

  so $F \otimes_\mathbb{Q} \mathbb{R}$ must be a product of copies of $\mathbb{C}$ and F is a CM-field.

  Take $F_1 = F_2 = F$: then on one hand $\mathfrak{h} \subseteq \mathfrak{su}(V/D) \subseteq \mathfrak{su}(V/F_1)$, and on the other hand $F_2 \supseteq E = F_2^+$.

- if A is of type IV, take a CM quadratic field k such that $\mathrm{Lie}(A)$ becomes a free module over $k \otimes \mathbb{C}$. Then Lemma 8.1.2 implies that we can take $F_1 = k$, and clearly we can take $F_2 = D$.

Lemma 8.1.8. *In the situation of the Theorem, the Lie algebras $\mathfrak{u}(V/F_1)$ and $\mathfrak{u}(V/F_2)$ have the same rank.*

Proof. Note that $\mathfrak{u}(V/F_i)$ is the Lie algebra of $\mathrm{Res}_{F_i/\mathbb{Q}}(\mathfrak{u}(V,\psi))$, so

$$\mathfrak{u}(V/F_i)_{\overline{\mathbb{Q}}} \cong (\mathfrak{u}(V/F_i))_{\overline{\mathbb{Q}}} \cong \left(\mathfrak{u}_{F_i}(V)^{[F_i:\mathbb{Q}]}\right)_{\overline{\mathbb{Q}}}.$$

The rank of $\mathfrak{u}_{F_i}(V)$ is $\frac{1}{2}\dim_{F_i}(V) = \frac{1}{2}\frac{\dim_{\mathbb{Q}} V}{[F_i:\mathbb{Q}]} = \frac{2\dim(A)}{2[F_i:\mathbb{Q}]}$, hence the rank of $\mathfrak{u}(V/F_i)$ is

$$\mathrm{rk}\left(\mathfrak{u}(V/F_i)_{\overline{\mathbb{Q}}}\right) = \mathrm{rk}\left(\mathfrak{u}_{F_i}(V)^{[F_i:\mathbb{Q}]}\right) = [F_i:\mathbb{Q}]\frac{2\dim(A)}{2[F_i:\mathbb{Q}]} = \dim(A),$$

independently of i. □

Let $V = H_1(A,\mathbb{Q})$ and $m = \dim_{F_1} V = \dfrac{2\dim(A)}{[F_1:\mathbb{Q}]}$. Consider the space L of F_1-Weil classes on A, namely

$$L := \bigwedge_{F_1}^{m} H^1(A,\mathbb{Q}).$$

L clearly has dimension one over F_1, so it is not the zero subspace. On the other hand, we are going to show that every element in L (except for 0) is an exceptional Hodge class.

- *L consists of Hodge classes*: to check that something is a Hodge class we only need to show that it is fixed by the action of the Hodge group. Note here that, as $F_1 \subseteq \mathrm{End}^0(A)$, the elements of F_1 commute with the action of the Hodge group, so in particular the Hodge group preserves the structure of F_1-vector space on V.

  An element $h \in \mathfrak{h}$ acts on L as $-tr_{F_1}(h)$, the trace of h thought of as a F_1-endomorphism of V: to see this, simply note that an element g of the Hodge group acts on $H^1(A,\mathbb{Q}) = H_1(A,\mathbb{Q})^*$ through g^{-1}, so it acts on the one-dimensional space L through $\det_{F_1}(g^{-1}) = \det_{F_1}(g)^{-1}$ (essentially by the very definition of the determinant in terms of exterior powers), so by differentiation the Lie algebra acts through $-tr_{F_1}(g)$. By construction, though, h belongs to $\mathfrak{su}(V/F_1)$, all of whose elements have trace zero, so the Hodge Lie algebra acts trivially on L and every element of L is a Hodge class.

- *All the elements of L are exceptional*: we start by showing that the action of $\mathfrak{u}(V/F_2)$ is trivial on divisor classes (and hence on the ring they generate).

  To this end, start by representing a divisor class as an alternating form $\delta(v,w)=\varphi(ev,w)$ for a certain $\dagger$-symmetric e. Thanks to Propositions 3.4.3 and 3.4.1 we can write

  $$\delta(v,w)=\varphi(ev,w)=tr_{F_2/\mathbb{Q}}\left(\psi(ev,w)\right)$$

  for a certain $\dagger$-symmetric e and an F_2-bilinear form ψ. By construction, $e=e^\dagger$ implies $e\in F_2$, so, by F_2-bilinearity we get

  $$\delta(v,w)=tr_{F_2/\mathbb{Q}}\left(e\psi(v,w)\right),$$

  whence the action of $h\in u\left(V/F_2\right)$ on δ is given by

  $$h\cdot\delta(v,w)=\delta(hv,w)+\delta(v,hw)=tr_{F_2/\mathbb{Q}}\left(e\left(\psi(hv,w)+\psi(v,hw)\right)\right),$$

  and since ψ is h-invariant by definition we get

  $$\left(\psi(hv,w)+\psi(v,hw)\right)=0\Rightarrow h\cdot\delta(v,w)=0,$$

  so that δ is h-invariant, too.

  Suppose now that Δ lies in the intersection $L\cap\mathscr{D}^\bullet(A)$. For every $h\in u\left(V/F_2\right)\subseteq u(V/F_1)$ we then have

  $$0=h\cdot\Delta=-tr_{F_1}(h)\Delta,$$

  so, if there is at least one such Δ that is non-trivial, we get $tr_{F_1}(h)=0$ for every $h\in\mathfrak{u}\left(V/F_2\right)\subseteq\mathfrak{u}\left(V/F_1\right)$, hence $\mathfrak{u}\left(V/F_2\right)\subseteq\mathfrak{su}\left(V/F_1\right)$. But this is absurd, since

  $$\operatorname{rank}\left(\mathfrak{su}\left(V/F_1\right)\right)<\operatorname{rank}\left(\mathfrak{u}\left(V/F_1\right)\right)=\operatorname{rank}\left(\mathfrak{u}\left(V/F_2\right)\right),$$

  where the last equality holds by Lemma 8.1.8. This contradiction shows that $\mathscr{D}^\bullet$ and L have trivial intersection, so every non-zero class in L is exceptional.

Finally, consider the case where A is defined over a number field K. Fix an embedding $\tau:K\hookrightarrow\mathbb{C}$ a consider the comparison isomorphism

$$H_1(A_\tau,\mathbb{Q})\otimes\mathbb{Q}_\ell\cong V_\ell(A).$$

As $\mathfrak{h}_\ell\subseteq\mathfrak{h}\otimes\mathbb{Q}_\ell$, the image in $V_\ell(A)$ of a Hodge class is a Tate class; moreover, from $\mathscr{D}^\bullet(A)\otimes\mathbb{Q}_\ell\cong\mathscr{D}^\bullet_\ell(A)$ it follows that it is an exceptional Tate class. $\square$

8.2 Exceptional classes: necessary condition

We break down the proof of the other implication in Theorem 8.0.5 in steps, one for each type in the Albert classification.

8.2.1 Type I

The endomorphism algebra of A is a totally real field E of degree $d|4$ over $\mathbb{Q}$. Two cases present themselves: $d = 1$ or $d > 1$.

In the second case, the results of Section 6.5 apply (since $\dim(A)/d \leq 2$), so the Hodge group of A coincides with $L(A)$ and Theorem 5.1.13 shows that the Hodge conjecture is true not only for A, but for its powers as well. Since we have analogous results also in the ℓ-adic setting, the same exact line of reasoning yields the Tate conjecture for A and all of its powers.

On the other hand, if $d = 1$, a slightly different method must be used: let $\overline{\mathbf{Q}}$ be a fixed algebraic closure of $\mathbf{Q}$ (note that in the ℓ-adic case we are *not* taking the completion). Extending scalars to $\overline{\mathbf{Q}}$ we get

$$\mathrm{End}_{\mathfrak{h}_{\overline{\mathbf{Q}}}}(V_{\overline{\mathbf{Q}}}) = \mathrm{End}(V)^{\mathfrak{h}} \otimes_{\mathbb{Q}} \overline{\mathbf{Q}} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \overline{\mathbf{Q}} \cong \overline{\mathbf{Q}},$$

so V is an absolutely irreducible representation of $Hg(A)$, and moreover $\mathfrak{h}$ is semisimple, since A is of type I. Suppose $\mathfrak{h}_{\overline{\mathbf{Q}}}$ is simple. Then $V_{\overline{\mathbf{Q}}}$ (as a $\mathfrak{h}_{\overline{\mathbf{Q}}}$-module) is faithful, irreducible, 8-dimensional, symplectic and minuscule: Theorem 2.4.6 implies that $\mathfrak{h}_{\overline{\mathbf{Q}}} \cong \mathfrak{sp}_{8,\overline{\mathbf{Q}}}$, so it is immediate to check that H must coincide with the Lefschetz group and the Hodge and Tate conjecture hold. If, on the other hand, $\mathfrak{h}_{\overline{\mathbf{Q}}}$ is not simple, write $\mathfrak{h}_1 \times \cdots \times \mathfrak{h}_l$ for its decomposition in simple factors and $V_{\overline{\mathbf{Q}}} \cong \boxtimes_{i=1}^{l} W_i$ for the corresponding decomposition of $V_{\overline{\mathbf{Q}}}$. The representation is faithful, so for each $i = 1, \cdots, l$ we have $\dim(W_i) > 1$; on the other hand, the dimension of each W_i divides 8, so - since we are assuming that there are at least 2 irreducible factors - one among the W_i's (say W_1) is 2-dimensional. By faithfulness, $\mathfrak{h}_1 \hookrightarrow \mathfrak{sl}(W_1) \cong \mathfrak{sl}_{2,\overline{\mathbf{Q}}}$, which in turn implies both $\mathfrak{h}_1 \cong \mathfrak{sl}_{2,\overline{\mathbf{Q}}}$ and $W_1 \cong \mathrm{Std}$, the standard representation of $\mathfrak{sl}_{2,\overline{\mathbf{Q}}}$. The remaining factors $W := W_2 \boxtimes \cdots \boxtimes W_l$ form a 4-dimensional vector space on which $\mathfrak{h}_2 \times \cdots \times \mathfrak{h}_l$ acts faithfully and preserving an orthogonal form (since the tensor product of W with W_1, which is a symplectic representation, must again yield a symplectic representation): by definition, this action factors through the standard representation of $\mathfrak{so}_{4,\overline{\mathbf{Q}}} \cong \mathfrak{sl}_{2,\overline{\mathbf{Q}}} \times \mathfrak{sl}_{2,\overline{\mathbf{Q}}}$. This forces $l = 2$ or 3, and - if we had $l = 2$ - the representation W would be irreducible, orthogonal and minuscule, which is impossible for $\mathfrak{sl}_{2,\overline{\mathbf{Q}}}$, so $l = 3$, and $V_{\overline{\mathbf{Q}}} \cong \mathrm{Std} \boxtimes \mathrm{Std} \boxtimes \mathrm{Std}$ (the representations of $\mathfrak{sl}_2$ are classified by their dimension, and we have just shown that we have three irreducible factors of dimension 2).

We now extend scalars to $\mathbf{C}$ and remark that in order to show that the Hodge (resp. Tate) conjecture holds for A it suffices to prove that the spaces

$$H^4(A,\mathbb{C})^{\mathfrak{h}_\mathbb{C}} \text{ (geometric case)}, \quad \left(\bigwedge^4 V^*_{\overline{\mathbf{C}}}\right)^{\mathfrak{h}_\mathbf{C}} \text{ (ℓ-adic case)}$$

are one-dimensional. This is now a matter of explicit computations: we can reduce to the second case by noticing that

$$H^4(A,\mathbb{C})^{\mathfrak{h}_\mathbb{C}} \cong \left(\bigwedge^4 V^*_\mathbb{C}\right)^{\mathfrak{h}_\mathbb{C}},$$

and we know the possibilities for $\mathfrak{h}_\mathbf{C}$ and for its action on $V_{\overline{\mathbf{C}}}$: either $\mathfrak{h}_\mathbf{C}$ is $\mathfrak{sp}_{8,\mathbf{C}}$ acting through its standard representation, or $\mathfrak{h}_\mathbf{C}$ is $\mathfrak{sl}_{2,\mathbf{C}}^3$ acting through $\mathrm{Std}^{\boxtimes 3}$ (these follow immediately from our computations over $\overline{\mathbf{Q}}$ by extending scalars).

Let us do the explicit computations for the first case: the standard representation $V^* \cong V$ is then defined by the highest weight ω_1 (cf. the table of minuscule weights). It is convenient to embed the root system in $\mathbb{R}^4$, taking as roots the vectors $\{\pm e_i \pm e_j, 2e_i\}_{1\leq i,j\leq 4}$, where e_i denotes the canonical base of $\mathbb{R}^4$. With this description, it is apparent that the Weyl group is the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes S_4$, where the factors act on the coordinates (respectively) by changing signs and by permutations. As the weight is minuscule, all the other weights appearing in the standard representation can be obtained from $\omega_1 = e_4$ as Weyl-conjugates, so we immediately see that the set of weights of the standard representation is $\{\pm e_i\}_{i=1,\cdots,4}$. Consequently, the weights in $\bigwedge^4 V$ are of the form $w = w_1 + w_2 + w_3 + w_4$, where the w_i's are pairwise distinct weights of the standard representation. Depending on whether or not we have $w_i = -w_j$ for certain pairs of indices we get:

- a set X_1 of weights of the form $\pm e_1 \pm e_2 \pm e_3 \pm e_4$, each with multiplicity one, for a total of 16 weights, all conjugated under the Weyl group;
- two orbits (under the Weyl group) X_2^1, X_2^2 of weights of the form $\pm e_i \pm e_j$ with $i \neq j$, each comprising 24 elements;
- the weight $w = 0$ with multiplicity six.

Weyl's dimension formula allows us to compute the dimension of the representations associated to the highest weights $e_1 + e_2 + e_3 + e_4$ and $e_1 + e_2$, that turn out to be 42 and 27 respectively. As the set of weights of an irreducible representation is stable under the action of the Weyl group, we see that the only possibility for the (multi)set of weights of $V(e_1+e_2+e_3+e_4)$ is $X_1 \cup X_2^1 \cup 2\{0\}$, and similarly for $V(e_1+e_2)$ is $X_2^2 \cup 3\{0\}$; it follows that

$$\Lambda^4(\mathrm{Std}) \cong V(e_1+e_2+e_3+e_4) \oplus V(e_1+e_2) \oplus V(0),$$

so the space of $\mathfrak{h}$-invariants in $\Lambda^4(\mathrm{Std})$ is of dimension one, as claimed.

If $\mathfrak{h}_{\mathbf{C}} \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$, similar computations allow us to conclude

$$\dim \left(\bigwedge^4 V_{\mathbf{C}}^* \right)^{\mathfrak{h}_{\mathbf{C}}} = 1,$$

so the Hodge and Tate conjectures hold for A, even though Theorem 5.1.13 does not apply.

Remark 8.2.1. An example by Mumford ([Mum69]) shows that there exist simple Abelian fourfolds with endomorphism algebra $\mathbb{Z}$ and Hodge group isomorphic to a $\mathbb{Q}$-form of $\mathfrak{sl}_2^3$.

Moreover, note that the Mumford-Tate conjecture is not known for simple fourfolds of type $I(1)$ (although the Hodge and Tate conjectures both follow from the above arguments): more precisely, it has not yet been proven that if $\mathrm{End}^0(A)$ is $\mathbb{Z}$ and the Hodge group is the full symplectic group $Sp(4)$, then $\mathfrak{h}_\ell$ is $\mathfrak{sp}_{4,\mathbb{Q}_\ell}$.

When the Hodge group is a $\mathbb{Q}$-form of SL_2^3, on the other hand, Deligne's theorem 5.3.6 shows that $\mathfrak{h}_\ell$ is forced to be a $\mathbb{Q}_\ell$-form of $\mathfrak{sl}_2^3$.

8.2.2 Type II

In this case $e = [E : \mathbb{Q}]$ is either 1 or 2, and in both cases we have $\frac{\dim(A)}{e} = 2h$ with $h \leq 2$, so our general results (Section 6.6) apply and yield that the Hodge group of A satisfies the hypothesis of Theorem 5.1.13, whence the Hodge conjecture is true for A and all of its powers.

Analogously, the same results apply in the ℓ-adic setting and the Tate conjecture holds for both A and its powers.

8.2.2.1 A simple proof of a special case

We sketch a different proof of the above result in the case $e = 2$. Essentially, all we need to do is describe $\mathfrak{u}(V/D)$ and calculate its dimension as $\mathbb{Q}$-vector space. Since both V and D are of dimension 8 over $\mathbb{Q}$ and every D-module is free, we conclude that $V \cong D$ as D-modules.

Let ρ be an automorphism of $V = D$ commuting with the left action of D. Then $\varphi(d) = \varphi(d \cdot 1) = d\varphi(1)$, so φ is the right multiplication by $\varphi(1) \in D$. For $d = \varphi(1)$ denote ρ_d this morphism.

A D-skew-Hermitian ψ is clearly of the form $(d_1, d_2) \mapsto d_1 a d_2^\dagger$ for a certain non-zero element $a \in D$ with $a^\dagger = -a$; it follows that an endomorphism ρ_d of V lies in $\mathfrak{u}_D(V, \psi)$ if and only if $da + ad^\dagger = 0 \Leftrightarrow da - a^\dagger d^\dagger = 0 \Leftrightarrow da = (da)^\dagger$.

By the Skolem-Noether theorem the Rosati involution is related to the canonical involution

$$e \mapsto \tilde{e} := Trd_{D/E}(e) - e$$

by $d^\dagger = b^{-1}\tilde{d}b$ for a certain non-zero $b \in D$. In fact, from the equality $a^\dagger = -a$ follows $b = a$, so that $\rho_d \in \mathfrak{u}_D(V, \psi)$ is equivalent to $da = (da)^\dagger = a^\dagger d^\dagger = -\tilde{d}a$, which in turn is equivalent to

$$d = -\left(Trd_{D/E}(d) - d\right).$$

We deduce that $\rho_d \in \mathfrak{u}_D(V, \psi)$ if and only if $Trd_{D/E}(d) = 0$, so we see that the isomorphism (of $\mathbb{Q}$-vector spaces)

$$\begin{array}{ccc} D & \xrightarrow{\sim} & \mathrm{End}_D(V) \\ d & \mapsto & \rho_d \end{array}$$

restricts to an isomorphism $\left\{d \in D | Trd_{D/E}(d) = 0\right\} \xrightarrow{\sim} \mathfrak{u}_D(V, \psi)$. Now D is $\mathbb{Q}$-vector space of dimension 8, and the equation $Trd_{D/E}(d) = 0$ imposes two (independent) $\mathbb{Q}$-linear conditions, so $\mathfrak{u}_D(V, \psi)$ is of dimension 6 over $\mathbb{Q}$. Extending scalars to $\mathbb{C}$, we get that the Lie algebra $\mathfrak{h}_\mathbb{C}$ is contained in a six-dimensional vector space over $\mathbb{C}$; now $\mathfrak{h}_\mathbb{C}$ is semisimple (as A is of type two), hence it admits a decomposition $\mathfrak{h}_\mathbb{C} \cong \prod_{i=1}^t \mathfrak{h}_i$, where each $\mathfrak{h}_i$ is a simple Lie algebra. We have an obvious constraint on the algebras $\mathfrak{h}_i$, namely the total dimension $\sum_{i=1}^t \dim_\mathbb{C} \mathfrak{h}_i$ must not exceed 6; in particular, the dimension of each simple factor must not exceed 6. The standard classification results for simple Lie algebras over $\mathbb{C}$ imply that the only simple Lie algebra of dimension at most 6 is $\mathfrak{sl}_2$, which is of dimension 3, so the only possibilities for $\mathfrak{h}_\mathbb{C}$ are $\mathfrak{sl}_2$ and $\mathfrak{sl}_2 \times \mathfrak{sl}_2$.

The possibility $\mathfrak{h}_\mathbb{C} \cong \mathfrak{sl}_2$ can be easily excluded, for example by observing that the decomposition of $V_\mathbb{C}$ as a product of 4 modules of dimension 2 would imply $\mathrm{End}(V_\mathbb{C})^{\mathfrak{sl}_2} \cong M_4(\mathbb{C})$, since there is only one 2-dimensional module over $\mathfrak{sl}_2$, up to isomorphism.

8.2.3 Type III

In this case we already know from Theorem 5.1.13 that we cannot expect the Hodge (resp. Tate) conjecture to hold in its strong form $\mathscr{D}^k(A^n) = \mathscr{B}^k(A^n)$ for every choice of natural numbers k, n. In fact, Theorem 8.1.1 implies that the space of Hodge classes of degree 4 is not generated by divisor classes, but we still get some positive results (note that the following is the only case we have to treat, since the results of Shimura collected in Proposition 7.1.2 exclude the case III(2)):

Proposition 8.2.2. *Let A be a 4-dimensional simple Abelian variety over $\mathbb{C}$ (resp. a number field K). Suppose that A is of type III, so its endomorphism algebra* $\operatorname{End}^0(A)$ *is a totally definite quaternion algebra D over $\mathbb{Q}$. Then*

- *$\mathfrak{h}$ is the centralizer of D in $\mathfrak{sp}(V,\varphi)$ (resp. $\mathfrak{h}_\ell$ is the centralizer of D_ℓ in $\mathfrak{sp}(V_\ell,\varphi_\ell)$)*
- *$\mathfrak{h}$ is a $\mathbb{Q}$-form of $\mathfrak{so}_4$ (resp. a $\mathbb{Q}_\ell$-form of $\mathfrak{so}_4$), so the Mumford-Tate conjecture holds for A*
- *The space $\mathscr{B}^2(A)$ (resp. $\mathscr{T}_\ell(A)$) is 6-dimensional, $\mathscr{D}^2(A)$ (resp. $\mathscr{D}^2_\ell(A)$) is 1-dimensional and $\mathscr{B}^2(A)=\mathscr{D}^2(A)+\mathscr{W}(A)$ (resp. $\mathscr{T}^2_\ell(A)=\mathscr{D}^2_\ell(A)+\mathscr{W}_\ell(A)$)*

Proof. By computing $\operatorname{End}(V_{\mathbf{C}})^{\mathfrak{h}_{\mathbf{C}}} \cong D\otimes\mathbf{C}\cong M_2(\mathbf{C})$ we find that $V_{\mathbf{C}}$ is isomorphic to the direct sum of two copies of an irreducible module W. Fix an isomorphism $V_{\mathbf{C}}\cong W\oplus W$.

We now want to better understand the bilinear form $\varphi_{\mathbf{C}}$ induced on $W\oplus W$ by the polarization on A. The space $\left(\bigwedge^2 V_{\mathbf{C}}\right)^{\mathfrak{h}_{\mathbf{C}}}\cong\mathscr{D}^1(A)\otimes\mathbf{C}$ is 1-dimensional: Lefschetz's Theorem identifies this space to the space of divisors on A, and Theorem 3.4.4 then yields that its dimension is the same as that of Rosati-invariant endomorphisms of A. Up to extension of scalars to $\mathbb{R}$, the Rosati involution is the standard involution on the quaternions, whose set of fixed points has dimension $\frac{1}{4}\dim_{\mathbb{R}}\mathbb{H}=1$. The equality

$$\left(\bigwedge^2(W\oplus W)\right)^{\mathfrak{h}}\cong\left(\bigwedge^2 W\right)^{\mathfrak{h}_{\mathbf{C}}}\oplus\left(\bigwedge^2 W\right)^{\mathfrak{h}_{\mathbf{C}}}\oplus(W\otimes W)^{\mathfrak{h}_{\mathbf{C}}}$$

then shows that $\left(\bigwedge^2 W\right)^{\mathfrak{h}_{\mathbf{C}}}$ must be trivial (otherwise we would have at least a *two*-dimensional invariant subspace), so $(W\otimes W)^{\mathfrak{h}_{\mathbf{C}}}$ is of dimension 1, which means that W is self-dual; as $\left(\bigwedge^2 W\right)^{\mathfrak{h}_{\mathbf{C}}}=0$, W cannot be symplectic, so it is orthogonal.

Choose a (nontrivial) $\mathfrak{h}_{\mathbf{C}}$-invariant orthogonal form α on W. Then the formula

$$\tilde{\varphi}\left(x_1\oplus x_2,y_1\oplus y_2\right)=\alpha(x_1,y_2)-\alpha(x_2,y_1)$$

defines a nontrivial, symplectic, $\mathfrak{h}_{\mathbf{C}}$-invariant bilinear form on $V_{\mathbf{C}}$. Since, as we have shown, the space of such forms is 1-dimensional, $\tilde{\varphi}$ must agree with $\varphi_{\mathbf{C}}$ up to multiplication by constants, so (rescaling α if necessary) we can assume $\varphi_{\mathbf{C}}=\tilde{\varphi}$. In particular, this implies $\mathfrak{h}_{\mathbf{C}}\subseteq\mathfrak{so}(W,\alpha)\cong\mathfrak{so}_4$.

We want to show that the inclusion is in fact an equality. Suppose by contradiction that $\mathfrak{h}_{\mathbf{C}}$ were smaller. As $\mathfrak{so}_4\cong\mathfrak{sl}_2\times\mathfrak{sl}_2$ is of rank 2, this would force the rank of $\mathfrak{h}_{\mathbf{C}}$ to be 1, and since $\mathfrak{h}$ is semisimple this would imply $\mathfrak{h} \cong \mathfrak{sl}_2$. But the irreducible representations of $\mathfrak{sl}_2$ are classified by their dimension, so W would be forced to be $\mathrm{Sym}^3(\mathrm{Std})$, which is a symplectic (hence non-orthogonal) representation, contradiction. We can therefore conclude that $\mathfrak{h}$ is a $\mathbf{Q}$-form of $\mathfrak{so}_4$, which in particular shows the Mumford-Tate conjecture for A.

We now turn to computing all the spaces involved in our statement. Let us start by describing the centralizer of $\mathrm{End}^0(A)$ in $\mathfrak{sp}(V,\phi)$. As this centralizer always contains $\mathfrak{h}$, it is enough to show that it has the same dimension as $\mathfrak{h}$, and to do this we can extend scalars to the algebraic closure. Let B be a linear transformation in $\mathfrak{sp}(V_{\mathbf{C}},\phi_{\mathbf{C}})$, and suppose that it commutes with $\mathrm{End}^0(A)\otimes\mathbf{C} \cong M_2(\mathbf{C})$. Then B commutes with both the projection on the first factor W and the swapping of the two factors W, which forces B to act as $B(w_1 \oplus w_2) = \tilde{B}w_1 \oplus \tilde{B}w_2$ for a certain endomorphism $\tilde{B}$ of W. Since B also preserves $\varphi_{\mathbf{C}}$ we immediately deduce

$$\alpha(x_1,y_2) = \varphi_{\mathbf{C}}(x_1\oplus 0, 0\oplus y_2) = \varphi_{\mathbf{C}}(B(x_1\oplus 0), B(0\oplus y_2)) = \alpha\left(\tilde{B}x_1, \tilde{B}y_2\right),$$

so $\tilde{B} \in \mathfrak{so}(W,\alpha)$. It follows that the extension to $\mathbf{C}$ of $\mathfrak{h}$ and of the centralizer of D inside $\mathfrak{sp}(V,\phi)$ agree, so these two spaces coincide, as we wanted to show.

The space $\mathscr{B}^2(A)$ of Hodge (resp. Tate) classes is 6-dimensional: as usual, being only interested in dimensions, we can extend scalars to an algebraic closure; W is then isomorphic to the standard representation of $\mathfrak{sl}_4$, and we need to compute the dimension of the invariant subspace in

$$\bigwedge^4 (\mathrm{Std} \oplus \mathrm{Std}),$$

which can be done by hand, keeping track of the weights that appear in the various exterior powers, or with the aid of a computer package (such as LiE).

We already remarked that $\mathscr{D}^1(A)$ is one-dimensional, hence the same holds for $\mathscr{D}^2(A)$. In order to better understand the structure of $\mathscr{B}^2(A)\otimes\mathbf{C}$ we exploit the fact that it is equipped with actions of both $\mathfrak{h}_{\mathbf{C}}$ and D^*, and the two commute. The derived subgroup $(D^*)'$ of D^* is simply the group of its elements of norm 1; regard it as an algebraic group over $\mathbb{Q}$ and let $\mathfrak{d}$ be its Lie algebra. $(D^*)'$ is semisimple and its Lie algebra is of dimension 3 (the Lie algebra of D^* is simply D, and $\mathfrak{d}$ is cut inside D by a single equation), so $\mathfrak{d}$ is a $\mathbb{Q}$-form of $\mathfrak{sl}_2$.

Observe now that V is free of rank 2 over D, so $\mathrm{End}(V)^{\mathfrak{d}} \cong M_2(D)$ and therefore

$$\mathrm{End}(V\otimes\mathbf{C})^{\mathfrak{d}_{\mathbf{C}}} \cong M_2(D)\otimes\mathbf{C} \cong M_2(M_2(\mathbf{C})) \cong M_4(\mathbf{C}),$$

hence $V_{\mathbf{C}}$ is, as a $\mathfrak{d}_{\mathbf{C}}$-module, isomorphic to the direct sum of four copies of the standard representation of $\mathfrak{sl}_{2,\mathbf{C}} \cong \mathfrak{d}_{\mathbf{C}}$. Let us fix the Cartan subalgebra of $\mathfrak{sl}_{2,\mathbf{C}}$ given by the diagonal matrices $\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$. Let ν be the character sending $\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$ to t. Then the standard representation of $\mathfrak{sl}_{2,\mathbf{C}}$ is the sum of two 1-dimensional character spaces, one corresponding to ν and one to $-\nu$. Consequently, $V_{\mathbf{C}}$ decomposes as the sum of two weight spaces $V_\nu, V_{-\nu}$, each of dimension 4.

We now remark that, since the actions of $\mathfrak{h}_{\mathbf{C}}$ and $\mathfrak{d}_{\mathbf{C}}$ on V commute, V_ν is a $\mathfrak{h}_{\mathbf{C}}$-submodule of $V_{\mathbf{C}}$ (and in fact it can be identified to one of the two copies of W, since these are the only two 4-dimensional submodules of V with respect to the action of $\mathfrak{h}_{\mathbf{C}}$). It follows that the 1-dimensional subspace

$$\bigwedge^4 V_\nu \subset \bigwedge^4 V$$

is a $\mathfrak{h}_{\mathbf{C}}$-submodule.

On the other hand, the Cartan subalgebra we have chosen acts on it via the character 4ν, so - as a module under $\mathfrak{d}_{\mathbf{C}}$ - $\left(\bigwedge^4 V\right)^{\mathfrak{h}_{\mathbf{C}}} = \mathscr{B}^2(A) \otimes \mathbf{C}$ contains at least an irreducible submodule of dimension 5. We also know that $\mathscr{D}^2(A) \otimes \mathbf{C}$ is a 1-dimensional $\mathfrak{d}_{\mathbf{C}}$-submodule, so - by comparing dimensions - we finally get the structure of $\mathscr{B}^2(A) \otimes \mathbf{C}$ as a module under the action of $\mathfrak{d}_{\mathbf{C}}$:

$$\mathscr{B}^2(A) \otimes \mathbf{C} \cong \operatorname{Sym}^0(\operatorname{Std}) \oplus \operatorname{Sym}^4(\operatorname{Std}),$$

where the trivial subrepresentation $\operatorname{Sym}^0(\operatorname{Std})$ can be identified with $\mathscr{D}^2(A) \otimes \mathbf{C}$.

Finally, $\mathscr{W}(A) \otimes \mathbf{C}$ is a $\mathfrak{d}_{\mathbf{C}}$-submodule of $\mathscr{B}^2(A) \otimes \mathbf{C}$ that is not contained in $\mathscr{D}^2(A) \otimes \mathbf{C}$ (the proof of Theorem 8.1.1 shows that it contains a subspace whose intersection with $\mathscr{D}^2(A)$ is trivial), so it must contain the irreducible submodule of dimension 5, and $\mathscr{W}$ and $\mathscr{D}^2(A)$ together generate $\mathscr{B}^2(A)$. □

8.2.4 Type IV

We start by fixing the notation we will use throughout the whole section. We denote by E a CM field of degree e over $\mathbb{Q}$, whose maximal totally real subfield will be denoted E_0. The unique nontrivial involution of E fixing E_0 will be denoted by $a \mapsto a'$. Let L be the normal closure of E in $\overline{\mathbb{Q}}$. By Corollaries 1.2.3 and 1.2.4 L is again a CM field, and complex conjugation on L (denoted ι) lies in the center of $\operatorname{Gal}(L/\mathbb{Q})$. If ρ is an L-valued map we also write $\overline{\rho}$ for $\iota \circ \rho$. Let furthermore e_0 be the degree of E_0 over $\mathbb{Q}$, $\rho_1, \ldots, \rho_{e_0}$ be the e_0 different embeddings of E_0 in L and $\sigma_1, \ldots, \sigma_{e_0}, \tau_1 = \iota \circ \sigma_1 = \overline{\sigma_1}, \ldots, \tau_{e_0} =$

$\iota \circ \sigma_{e_0} = \overline{\sigma_{e_0}}$ the $2e_0 = e$ embeddings of E in L. Fixing once and for all an embedding $L \hookrightarrow \mathbb{C}$ we identify embeddings of E in L and in $\mathbb{C}$. We also set $R := \operatorname{Hom}(E, \mathbb{C}) = \{\sigma_1, \ldots, \sigma_{e_0}, \tau_1, \ldots, \tau_{e_0}\}$ and $T_E := \operatorname{Res}_{E/\mathbb{Q}}(\mathbb{G}_{m,E})$; clearly T_E is a $\mathbb{Q}$-torus of rank $[E : \mathbb{Q}]$, whose character group is the free Abelian group on the set R.

Let Θ be the subgroup of $X^*(T_E)$ generated by the elements $\sigma_i + \iota \circ \sigma_i$ for i ranging from 1 to e_0: we have $\Theta \cong \bigoplus_{i=1}^{e_0} \mathbb{Z}(\sigma_i + \tau_i)$. Inside T_E lies the subtorus $U_E := \{a \in T_E | aa' = 1\}$, whose character group is

$$X^*(U_E) = \frac{X^*(T_E)}{\bigoplus_{i=1}^{e_0} \mathbb{Z}(\sigma_i + \tau_i)} = \frac{X^*(T_E)}{\Theta}.$$

Finally, if k is an imaginary quadratic field contained in E, we set

$$SU_{E/k} := \left\{x \in U_E | N_{E/k}(x) = 1\right\}.$$

The character group of $SU_{E/k}$ is the quotient of $X^*(U_E)$ by the submodule generated by $\sum_{\sigma \in \operatorname{Gal}(L/K)} \sigma[\rho]$ for $[\rho]$ varying in $X^*(U_E)$ (this follows immediately by differentiating the definition of the norm). It is easy to check that $SU_{E/k}$ is of (pure) codimension 1 inside U_E: it has codimension at most 1 by Krull's theorem (note that the condition on the norm can be expressed, for elements of U_E, by just one equation), and the equality can easily be shown at the level of Lie algebras.

Indeed, note that E_0 and k intersect only trivially, so

$$2[E_0 : \mathbb{Q}] \leq [E_0 k : \mathbb{Q}] \leq [E : \mathbb{Q}] = 2[E_0 : \mathbb{Q}]$$

and $E = E_0 k$.

We have the following simple description for the Lie algebras: on one hand, $\mathfrak{u}_E = \{a \in E | a + a' = 0\}$, and on the other

$$\mathfrak{su}_{E/k} = \left\{a \in \mathfrak{u}_E | \operatorname{tr}_{E/k}(a) = 0\right\};$$

write $E = E_0[i]$ with $i \in k$ and $a = e_1 + e_2 i$ ($e_1, e_2 \in E_0$) for a generic element of E. If we had the equality $SU_{E/k} = U_E$, then $a + a' = 0$ would imply $\operatorname{tr}_{E/k}(a) = 0$, but with the above notation the first condition is simply $e_1 = 0$, so we should have $\operatorname{tr}_{E/k}(e_2 i) = 0$ for every $e_2 \in E_0$, which clearly does not happen.

Summarizing, we have thus established the following

Proposition 8.2.3. *Notation as above.*

For every imaginary quadratic field k contained in E the torus

$$SU_{E/k} := \left\{x \in U_E | N_{E/k}(x) = 1\right\}$$

has codimension 1 in U_E and character group

$$X^*\left(SU_{E/k}\right) = \frac{X^*\left(U_E\right)}{\left\langle \sum_{\sigma\in \mathrm{Gal}(L/K)} \sigma[\rho] \right\rangle_{[\rho]\in X^*(U_E)}}$$

We have the following converse, which is the 'Key Lemma' of the last section of [MZ95]:

Lemma 8.2.4. *Suppose H is an algebraic torus of codimension 1 in U_E. Then there exists an imaginary quadratic field k contained in E such that $H = SU_{E/k}$.*

Proof. The inclusion $H \hookrightarrow U_E$ induces a surjection

$$\pi : X^*\left(U_E\right) \twoheadrightarrow X^*\left(H\right)$$

on character groups. Let Δ be the kernel of π. Character groups are free Abelian (of finite rank), so Δ itself has to be free, and is of rank one because H is of codimension 1 in U_E. Choose a generator δ of Δ.

The Galois module structure of the character groups also yields a map $\kappa : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Aut}\left(\Delta\right) \cong \{\pm 1\}$. Let I be the kernel of κ.

Observe that ι acts on $X^*\left(U_E\right)$ as multiplication by -1: indeed, $X^*\left(U_E\right)$ is generated by $\sigma_1, \ldots, \sigma_{e_0}$, and for every index i we have $[\iota\circ\sigma_i] = [\tau_i] = -[\sigma_i]$ in the quotient by Θ. In particular, ι acts as multiplication by -1 on Δ, so κ is surjective and I is normal of index 2 in $\mathrm{Gal}(L/\mathbb{Q})$. Let k be the field associated to I by Galois theory: it is a Galois extension of $\mathbb{Q}$ of degree 2, and - since it is not fixed by complex conjugation, which does not belong to I - it is an imaginary quadratic field.

We want to show that $H = SU_{E/k}$.

Write the generator δ of Δ as $\sum_{\rho\in R} c(\rho)\,[\rho] \mod \Theta$. The integer $c(\rho)$ is not well-defined, but the difference $c\left(\rho\right) - c\left(\overline{\rho}\right)$ is. Fix a $\rho_0 \in R$. By transitivity of the Galois action, for every $\rho_1 \in R$ there exists $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$ such that $\sigma\circ\rho_0 = \rho_1$, and $\sigma \circ \overline{\rho_0} = \overline{\sigma\circ\rho_0} = \overline{\rho_1}$, since ι belongs to the center of the Galois group. For any σ in the Galois group, thinking κ as a map with values in $\{\pm 1\}$, we get

$$\sigma(\delta) = \kappa(\sigma)\delta = \kappa(\sigma)\cdot \sum_{\rho\in R} c(\rho)\,[\rho] \mod \Theta,$$

and on the other hand

$$\sigma(\delta) = \sum_{\rho\in R} c(\rho)\,[\sigma\circ\rho] \mod \Theta.$$

Using the equality $\overline{\rho_1} = \sigma\circ\overline{\rho_0}$ and comparing the value of $c\left(\rho_1\right) - c\left(\overline{\rho_1}\right)$ in the two expressions for $\sigma\left(\delta\right)$ we get $\kappa(\sigma)\left(c\left(\rho_1\right) - c\left(\overline{\rho_1}\right)\right) = c\left(\rho_0\right) - c\left(\overline{\rho_0}\right)$, so there exists a non-negative integer m such that, for every $\rho \in R$, the absolute value of $c(\rho) - c(\overline{\rho})$ equals m. Note that $m \neq 0$, for otherwise δ would be zero in the quotient modulo Θ and π would be injective, which is not the case.

In order to identify k to a subfield of E we show that $k \subseteq \rho(E)$ for any embedding $\rho \in R$, and to this end it suffices to show that any automorphism $\sigma \in \mathrm{Gal}(L/\rho(E))$ fixes k. Suppose by contradiction $\sigma \circ \rho = \rho$ but $\sigma(\delta) = -\delta$. Then $c(\rho) - c(\rho') = -c(\rho) + c(\rho')$, so $m = 0$ and $\delta = 0$, which is absurd.

Finally, by construction (and Galois theory) $\mathrm{Gal}(L/k)$ is a quotient of $\ker \kappa$, so every σ in $\mathrm{Gal}(L/k)$ acts trivially on δ; it follows that

$$[L:k]\delta = \sum_{\sigma \in \mathrm{Gal}(L/k)} \sigma\delta = \sum_{\sigma \in \mathrm{Gal}(L/k)} \sigma \left(\sum_{\rho \in R} c(\rho)\,[\rho] \mod \Theta \right) =$$

$$= \sum_{\rho \in R} c(\rho) \left(\underbrace{\sum_{\sigma \in \mathrm{Gal}(L/k)} [\sigma \circ \rho] \mod \Theta}_{0} \right) = 0 \text{ in } X^*\left(SU_{E/k}\right),$$

since - as we already remarked - $X^*\left(SU_{E/k}\right)$ is the quotient of U_E by the subgroup generated by $\sum_{\sigma \in \mathrm{Gal}(L/k)} [\sigma \circ \rho] \mod \Theta$ for ρ varying in R. This implies that $[L:k]\delta$ belongs to the kernel of the projection

$$\pi_1 : X^*(U_E) \to X^*\left(SU_{E/k}\right),$$

so δ is a torsion element in the quotient $X^*\left(SU_{E/k}\right)$, which is free, so δ itself is mapped to zero by π_1. Hence $\ker \pi \subseteq \ker \pi_1$, and since both these groups are free of rank 1 and both quotients are free this must be an equality, as we wanted to show. □

From now on, we suppose that A is a simple Abelian fourfold of type IV defined over $\mathbb{C}$ (resp. a number field K).

Thanks to Proposition 7.1.2, the endomorphism algebra of A is a CM field E, whose degree e over $\mathbb{Q}$ has to divide 8 by the Albert classification. We therefore need to tackle three cases, corresponding to the three possible values of e_0: 1, 2 or 4.

8.2.4.1 Type IV(1,1)

Keeping all of the above notation, suppose $e_0 = 1$, so E is an imaginary quadratic field.

Then we have exactly two possibilities:

- E acts with multiplicities $\{n_\sigma, n_\tau\} = \{1,3\}$, in which case $\mathfrak{h} = \mathfrak{u}(V/E)$, so Theorem 5.1.13 (or its ℓ-adic analogue) applies and yields that the Hodge (resp. Tate) conjecture is true for all powers of A;
- E acts with multiplicities $n_\sigma = n_\tau = 2$, in which case $\mathfrak{h} = \mathfrak{su}(V/E)$ and the relevant spaces have the following dimensions:
  - $\mathscr{B}^2(A)$ is 3-dimensional
  - $\mathscr{D}^2(A)$ is 1-dimensional
  - $\mathscr{W}(A) \cong \bigwedge_E^4 V^*$ is 2-dimensional

  Moreover, $\mathscr{B}^2(A) \cong \mathscr{D}^2(A) \oplus \mathscr{W}(A)$.

Proof. By Proposition 7.1.2 we have $n_\sigma n_\tau \neq 0$, and since we know that $n_\sigma + n_\tau = 4$ it follows that the two above are the only possible cases.

Assume that $\{n_\sigma, n_\tau\} = \{1,3\}$: then this becomes an immediate application of Theorem 7.3.1.

Suppose, on the contrary, $n_\sigma = n_\tau = 2$. Then Lemma 8.1.2 ensures $\mathfrak{h} \subseteq \mathfrak{su}(V/E)$ (resp. $\mathfrak{h}_\ell \subseteq \mathfrak{su}(V_\ell/E_\ell)$).

We want to show that $\mathfrak{h}_\mathbf{C}$ is semisimple, in order to use our usual representation theoretic machinery.

From Lemma 3.2.9 we know that the center of $\mathfrak{h}$ is contained in the Lie algebra $\mathfrak{u}_E = \{e \in E | e + e' = 0\}$; on the other hand, $\mathfrak{h}^{ab} \subset \mathfrak{h} \subseteq \mathfrak{su}(V/E) \subseteq \{e \in \operatorname{End}_E(V) | \operatorname{tr}_E(e) = 0\}$. An element in the intersection is therefore a scalar acting on V with trace zero, hence it is zero: this shows that the Abelian part of $\mathfrak{h}$ is trivial and $\mathfrak{h}$ is semisimple. The same argument also works in the ℓ-adic case, where the conclusion of Lemma 3.2.9 is replaced by $\mathfrak{h}_\ell \subseteq E \otimes \mathbb{Q}_\ell$ (the proof being essentially identical, cf. [Chi90], Proposition 3.1.1).

The inclusion $\mathfrak{h} \subseteq \mathfrak{su}(V/E)$ implies $\mathfrak{h}_\mathbf{C} \subseteq \mathfrak{sl}_{4,\mathbf{C}}$, the latter acting on $V_\mathbf{C}$ as $\operatorname{Std} \oplus \operatorname{Std}^*$. Write $V_\mathbf{C} \cong V_1 \oplus V_2$ for the decomposition of $V_\mathbf{C}$ with respect to the action of $\mathfrak{sl}_4$: this is also a decomposition under the action of $\mathfrak{h}_\mathbf{C}$. The usual argument shows that V_1, V_2 are irreducible and non-isomorphic as $\mathfrak{h}$-modules:

$$\operatorname{End}(V_\mathbf{C})^{\mathfrak{h}} \cong \operatorname{End}(V)^{\mathfrak{h}} \otimes \mathbf{C} \cong E \otimes \mathbf{C} \cong \mathbf{C}^2,$$

where we use the fact that ℓ splits in E.

As V_2 is the dual representation of V_1 and by the above computation it is *not* isomorphic to it, we see that the action of $\mathfrak{h}_\mathbf{C}$ on V_1 must be irreducible, given by minuscule weights and of dimension 4, and on the other hand $\operatorname{rank} \mathfrak{h} \leq \operatorname{rank} \mathfrak{sl}_4 = 3$. Using the results collected in the table of Theorem 2.4.6 we find that $\mathfrak{h}_\mathbf{C} \cong \mathfrak{sl}_4$ (recall that the root systems A_3 and D_3 are actually isomorphic) and that V_1 is isomorphic to the standard representation of $\mathfrak{sl}_4$.

Remark 8.2.5. The proof of Theorem 7.4 in [MZ95] relies on the fact that the space $\left(\bigwedge^2 V_{\mathbf{C}}\right)^{\mathfrak{h}}$ is one-dimensional to deduce that V_1 is not self-dual. Though in this specific case the conclusion holds, taking $\mathfrak{h} = \mathfrak{so}_4$, $V_1 = \mathrm{Std}$ and $V_2 = \mathrm{Std}^* \cong \mathrm{Std}$ gives an example where $\left(\bigwedge^2 V_{\mathbf{C}}\right)^{\mathfrak{h}}$ is one-dimensional and V_1 is self dual.

This is easily checked, since

$$\begin{aligned}\bigwedge^2 (\mathrm{Std} \oplus \mathrm{Std}) &\cong \left(\bigwedge^2 \mathrm{Std}\right)^{\oplus 2} \oplus (\mathrm{Std} \otimes \mathrm{Std}) \\ &\cong \left(\bigwedge^2 \mathrm{Std}\right)^{\oplus 3} \oplus \mathrm{Sym}^2 (\mathrm{Std}),\end{aligned}$$

and the invariant subspace is of dimension one, coming from the single line of orthogonal forms in $\mathrm{Sym}^2 (\mathrm{Std})$.

By rank considerations we therefore have $\mathfrak{h} = \mathfrak{su}(V/E)$ (resp. $\mathfrak{h}_\ell = \mathfrak{su}(V_\ell/E_\ell)$ in the ℓ-adic case).

Everything else now follows from a direct computation: the dimension of the space of divisors is the same as that of $\left(\bigwedge^2 V_{\mathbf{C}}^*\right)^{\mathfrak{h}}$, which in turn is exactly the same as $\dim_{\mathbb{Q}} E_0 = 1$ (these being the Rosati-symmetric endomorphisms) by Theorem 3.4.4, so the space $\mathscr{D}^2(A)$ is 1-dimensional, while the space of invariants inside $\bigwedge^4 (V_1 \oplus V_1^*)$, which we write as

$$\bigwedge^4 V_1 \oplus \left(\bigwedge^3 V_1 \otimes V_1^*\right) \oplus \left(\bigwedge^2 V_1 \otimes \bigwedge^2 V_1^*\right) \oplus \left(V_1 \otimes \bigwedge^3 V_1^*\right) \oplus \bigwedge^4 V_1^*,$$

is 3-dimensional. This is perhaps most easily seen by exploiting the simmetry $\bigwedge^3 V_1 \cong V_1^*$:

- The first and last summand in the direct sum decomposition are trivial representations (there is no non-trivial morphism from a simple group to the 1-dimensional torus $\mathbb{G}_m$).
- The summand $\bigwedge^3 V_1 \otimes V_1^* \cong (V_1^*)^{\otimes 2}$ has no invariants, since the invariants in $(V_1^*)^{\otimes 2} \cong S^2 V_1 \oplus \bigwedge^2 V_1$ are precisely the $\mathfrak{h}$-equivariant morphisms between V_1 and V_1^*, which we know to be reduced to 0. The same argument also works for the next-to-last summand.
- Finally, $\bigwedge^2 V_1 \cong \bigwedge^2 V_1^*$ are both isomorphic to the self-dual (orthogonal) minuscule representation given by ω_2, so the space of its invariants is 1-dimensional, exactly because this is the necessary and sufficient condition in order for an irreducible representation to be self-dual.

$\mathscr{W}(A) = \bigwedge_E^4 V^*$ is of dimension 1 over E (being the fourth exterior power over E of a 4-dimensional space over E), so it is 2-dimensional over $\mathbb{Q}$. Since both $\mathscr{D}^2(A)$, of dimension 1, and $\mathscr{W}(A)$, of dimension 2, are contained in $\mathscr{B}^2(A)$ (whose dimension we computed to be 3) and their intersection is trivial because of the proof of Theorem 8.1.1, we finally get

$$\mathscr{B}^2(A) \cong \mathscr{D}^2(A) \oplus \mathscr{W}(A)$$

and we are done. □

8.2.4.2 Type IV(2,1)

In order to treat the ℓ-adic case we shall need a few arithmetic preliminaries.

Proposition 8.2.6. *Let u be algebraic over $\mathbb{Q}$, $u \notin \mathbb{Q}$.*

Let $S = \{u_1 = u, u_2, \dots, u_n\}$ be the set of Galois conjugates of u, and let Γ be the multiplicative subgroup of $\overline{\mathbb{Q}}^*$ generated by $u_1, \dots, u_n$. If Γ is free of rank 1, then $\mathbb{Q}(u)$ is a quadratic field.

Proof. Let γ be a generator of Γ and consider $F = \mathbb{Q}(\gamma)$. Clearly $\mathbb{Q}(u)$ is contained in $\mathbb{Q}(\gamma)$, so it is enough to show that F is a quadratic field.

We start by showing that F is Galois over $\mathbb{Q}$, that is, that all the Galois conjugates of γ are in F. Take any embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$. σ acts on the set S as a permutation: write $\sigma(u_i) = u_{\tau(i)}$, where $\tau \in \mathcal{S}_n$. By definition of Γ, there exist integers $m_1, \dots, m_n$ such that $\gamma = \prod_{i=1}^{n} u_i^{m_i}$: then

$$\sigma(\gamma) = \prod_{i=1}^{n} \sigma(u_i)^{m_i} = \prod_{i=1}^{n} u_{\sigma(i)}^{m_i} \in \Gamma,$$

and since every element of Γ is a power of γ we finally have $\sigma(\gamma) = \gamma^k$ for a certain $k \in \mathbb{Z}$, which shows that every Galois conjugate of γ belongs to F. Moreover, the same argument also shows that any $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ restricts to an automorphism of Γ, whence a map

$$\psi : \mathrm{Gal}(F/\mathbb{Q}) \to \mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}.$$

On the other hand, ψ is injective, because an element $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ is completely determined by $\sigma(\gamma)$; it follows that $\mathrm{Gal}(F/\mathbb{Q})$ has at most two elements, i.e. that $[F : \mathbb{Q}] \leq 2$. This forces $[\mathbb{Q}(u) : \mathbb{Q}] \leq [F : \mathbb{Q}] \leq 2$, so $\mathbb{Q}(u)$ is a quadratic field and coincides with $\mathbb{Q}(\gamma)$.

□

Proposition 8.2.7. *Let K be a number field, k a topological field and $\rho : \operatorname{Gal}(\overline{K}/K) \to GL(k)$ a continuous Galois representation that ramifies at most at finitely many primes of K.*

Then the set $\{\rho(\operatorname{Frob}_v) | v \text{ unramified}\}$ is a dense subset of the image of ρ.

Proof. In a profinite group G a subset X is dense if and only if for every finite quotient $\pi : G \twoheadrightarrow G_i$ the image of X through π equals G_i.

We apply the above to the case where $G = \rho\left(\operatorname{Gal}(\overline{K}/K)\right)$ and X is the set of Frobenius images: the finite quotients of G correspond to finite Galois extensions of K and, consequently, Chebotarev's density theorem implies that the image of X in any finite quotient is all of it. □

Proposition 8.2.8. *Let A be an Abelian variety of dimension g defined over the number field K, ℓ a rational prime and $\rho_\ell : \operatorname{Gal}\left(\overline{K}/K\right) \to \operatorname{End}(T_\ell(A))$ the continuous ℓ-adic Galois representation attached to A.*

Then, for every choice of ℓ outside a finite set, there exists a Frobenius element $\operatorname{Frob}_v \in \operatorname{Gal}\left(\overline{K}/K\right)$ such that

- *ℓ is unramified in K*
- *there is a place w of K above ℓ of good reduction for A*
- *the multiplicative subgroup of $\overline{\mathbb{Q}}^*$ generated by the eigenvalues of Frob_v (thought of as an endomorphism of the ℓ-adic Tate module) does not contain any nontrivial root of unity.*

Proof. We begin by fixing our notation. v will be a place of K above the rational prime p, which we will tacitly assume to be distinct from ℓ. The characteristic polynomial of a Frobenius element Frob_v acting on $T_\ell(A)$ is monic with integral coefficients and does not depend on the choice of ℓ (as long as $\ell \neq p$); let it be denoted by $\Phi_v(x)$. The splitting field of $\Phi_v(x)$ over $\mathbb{Q}$ will be denoted F_v, and we will write M_v for the subgroup of $\overline{\mathbb{Q}}^*$ generated by the eigenvalues of Frob_v acting on $T_\ell(A)$.

Clearly $\Phi_v(x)$ has degree $2g$, where g is the dimension of A, since this is the rank of $T_\ell(A)$. The degree $[F : \mathbb{Q}]$ is therefore uniformly bounded by a function of g (and we can take this function to be $(2g)!$). If a root of unity ζ_n of order n belongs to M_v, then it certainly belongs to F_v, so $\varphi(n) = [\mathbb{Q}(\zeta_n) : \mathbb{Q}] \leq [F_v : \mathbb{Q}] \leq (2g)!$ is bounded (independently of ℓ).

It is well known that, for a fixed k, the number of solutions to $\varphi(n) \leq k$ is finite, so there exists an integer M *independent of* ℓ such that any root of unity ζ in F_v satisfies $\zeta^M = 1$.

Let $\Psi(x)$ be the polynomial $\frac{x^M - 1}{x - 1}$ and Ω_K be the set of finite places of K.

Our admissible set of primes will be

$$\left\{ \ell \text{ rational prime} \,\middle|\, \begin{array}{c} \exists w \in \Omega_K, w|\ell, \text{ of good reduction for } A \\ \text{and } \ell \text{ does not divide } \Psi(1) \end{array} \right\}$$

Clearly there is only a finite number of primes that do not belong to the above set.

We want to explicitly determine a neighborhood U of the identity of $\mathrm{Gal}\left(\overline{K}/K\right)$ such that, for every Frobenius element Frob_v in U, M_v does not contain any nontrivial root of unity. This, together with the density of Frobenius elements in $\mathrm{Gal}\left(\overline{K}/K\right)$, would be enough to prove the proposition. Consider the continuous Galois representation

$$\rho_\ell : \mathrm{Gal}\left(\overline{K}/K\right) \to \mathrm{Aut}(T_\ell(A))$$

and identify $\mathrm{Aut}(T_\ell(A))$ with $\mathrm{GL}\left(\mathbb{Z}_\ell\right)$ through the choice of a basis. Let U_ℓ be the open neighborhood of the identity in $\mathrm{GL}\left(\mathbb{Q}_\ell\right)$ given by the matrices N such that $||N - id||_\ell < \ell^{-1}$, i.e. those matrices that can be written as $id + P$, where $P \in \mathcal{M}_{2g\times 2g}\left(\mathbb{Z}_\ell\right)$ has all its coefficients divisible by ℓ.

We set $U := \rho_\ell^{-1}\left(U_\ell\right)$. This is clearly an open neighbourhood of the identity, and we want to show that every Frobenius element F_v in U has the required properties. We argue by contradiction, supposing there is a nontrivial root of unity in M_v.

Let therefore Frob_v be in U and $\lambda_1, \cdots, \lambda_{2g} \in F_v$ be the eigenvalues of Frob_v. Note that each λ_i is an algebraic integer, so it belongs to $\mathcal{O}_{F_v}$. Choose a place $\mathfrak{l}$ of F_v above ℓ and let $\overline{\lambda}_1, \cdots, \overline{\lambda}_{2g}$ be the reductions of $\lambda_1, \cdots, \lambda_{2g}$ in $\mathcal{O}_{F_v}/\mathfrak{l}$.

We claim that $\overline{\lambda}_i = 1$. To see this, simply notice that the $\overline{\lambda}_i$'s are the eigenvalues of the reduction (modulo ℓ) of $\rho_\ell\left(\mathrm{Frob}_v\right)$ acting on $\mathbb{F}_\ell^{2g}$: indeed, taking reductions (modulo ℓ) and taking characteristic polynomials commute, so the eigenvalues of the reduction are the reductions of the eigenvalues. Note that since $F_v/\mathbb{Q}$ is Galois, all the places above ℓ are conjugated to each other, so everything is well-defined up to the action of the Galois group, and in particular the finite field $\mathcal{O}_{F_v}/\mathfrak{l}$ does not depend on our choice of $\mathfrak{l}$, up to isomorphism.

The definition of U_ℓ implies that $\rho_\ell\left(\mathrm{Frob}_v\right)$ reduces to the identity, so the reductions $\overline{\lambda}_i$ equal the eigenvalues of the identity, i.e. every $\overline{\lambda_i}$ equals 1.

Suppose now $\zeta = \lambda_1^{n_1} \cdots \lambda_{2g}^{n_{2g}} \in M_v$ is a nontrivial root of unity. ζ belongs to $\mathcal{O}_F$ (it belongs to F by construction and it certainly is an algebraic integer), so it makes sense to consider $\overline{\zeta}$, its reduction modulo $\mathfrak{l}$.

On one hand, the equation $\zeta = \lambda_1^{n_1} \cdots \lambda_{2g}^{n_{2g}} \in M_v$ implies

$$\overline{\zeta} = \prod_{i=1}^{2g} \overline{\lambda}_i^{n_i} = 1 \pmod{\mathfrak{l}}.$$

On the other hand, we know $\zeta^M = 1$, and since $\zeta \neq 1$ we also get $\Psi(\zeta) = 0$ (recall that $\Psi(x) = \dfrac{x^M - 1}{x-1}$); now $\Psi(x)$ is a polynomial with integral coefficients, so

$$0 = \overline{\Psi(\zeta)} = \overline{\Psi}\left(\overline{\zeta}\right) = \overline{\Psi}(1) = \overline{\Psi(1)} \pmod{\mathfrak{l}} :$$

as $\Psi(1)$ is a rational integer, this clearly implies $\Psi(1) \equiv 0 \pmod{\ell}$, which contradicts our choice of ℓ. This completes the proof. □

We are now in a position to prove the following

Proposition 8.2.9. *Notation as above. If A is of type IV(2,1), then $\mathfrak{h} \cong \mathfrak{u}_E(V,\psi)$ (resp. $\mathfrak{h} \cong \mathfrak{u}_{E_\ell}(V_\ell,\psi_\ell)$), so Theorem 5.1.13 applies and the Hodge (resp. Tate) conjectures holds for all the powers of A.*

Proof. Thanks to Theorem 5.3.5, and since we only need to establish results about ranks, we can choose ℓ to be a prime that splits completely in E and such that there is a place of E of good reduction for A of residue characteristic ℓ.

We have $n_{\sigma_1}+n_{\tau_1} = n_{\sigma_2}+n_{\tau_2} = 2$, and by Proposition 7.1.2 we can assume that

$$(n_{\sigma_1}, n_{\tau_1}) = (2,0), (n_{\sigma_2}, n_{\tau_2}) = (1,1). \quad (*)$$

We want to study the semisimple and the Abelian part of $\mathfrak{h}$ separately. Let $\mathfrak{c}$ be the center of $\mathfrak{h}$. Lemma 3.2.9 and its obvious ℓ-adic analogue imply that $\mathfrak{c}$ is contained in the 2-dimensional Lie algebra $\mathfrak{u}_E = \{a \in E | a' = -a\}$ (resp $\mathfrak{u}_{E_\ell}$).

Suppose $\mathfrak{c} = 0$. Then $\mathfrak{h}$ is semisimple and contained in $\mathfrak{su}(V/E)$, which is impossible, since the centralizer of $\mathfrak{su}(V/E)$ in $\operatorname{End}(V)$ is a quaternion algebra over E_0, while the centralizer of $\mathfrak{h}$ is the endomorphism algebra, i.e. E itself.

Suppose $\mathfrak{c}$ is of rank 1. In this case we really need to treat the geometric and ℓ-adic cases separately.

In the Hodge case, Key Lemma 8.2.4 yields the existence of an imaginary quadratic field k such that $\mathfrak{c} = \mathfrak{su}_{E/k}$, so $\mathfrak{hg} \subseteq \mathfrak{su}_{V/k}$ and using Lemma 8.1.2 we get a contradiction with $(*)$.

We now turn to the ℓ-adic case. Proposition 8.2.8 allows us to choose a Frobenius element $\operatorname{Fr}_v \in G_\ell(\mathbb{Q}_\ell)$ such that the multiplicative group generated by its eigenvalues does not contain any root of unity different from 1. Let q be the cardinality of the residue field at v and consider $\lambda := \operatorname{Fr}_v^2 / q$. The determinant of Fr_v acting on the ℓ-adic module of A is $q^{\dim(A)}$, so λ actually belongs to $H_\ell(\mathbb{Q}_\ell)$. Let $u = \det_{E_\ell} \lambda \in E_\ell \cong \mathbb{Q}_\ell^4$.

Note that we have an embedding $E \hookrightarrow E_\ell \cong \mathbb{Q}_\ell^4$ given, as usual, by

$$e \mapsto (\sigma_1(e), \tau_1(e), \sigma_2(e), \tau_2(e)),$$

and because of the compatibility of the Frobenius with the Galois action the determinant of λ lies in this copy of E inside E_ℓ. Moreover, the ℓ-adic logarithm of u does not vanish, since u is not divisible by ℓ. Note that we can suppose that $\log u$ belongs to $\mathfrak{h}_\ell$: the ℓ-adic logarithm maps a neighborhood of the identity of H_ℓ to $\mathfrak{h}_\ell$, so all we need to do is choose Fr_v sufficiently close to id, which is always possible.

Thinking $\log(u)$ as an E_ℓ-linear operator on V_ℓ we can write $\log(u) = \log\det\lambda = \operatorname{tr}\log(\lambda)$ (the latter being well-defined assuming we have chosen the Frobenius element sufficiently close to the identity); on the other hand, the trace operator is zero on the semi-simple part of $\mathfrak{h}_\ell$ and is multiplication by 2 on the Abelian part, since V_ℓ is free of rank 2 over E_ℓ. It follows that $\log(u)$ belongs to $\mathfrak{c}_\ell$, hence - assuming this is of rank 1 - $\mathfrak{c}_\ell = \mathbb{Q}_\ell \log(u)$.

$\mathfrak{c}_\ell$ is an algebraic Lie algebra (it is the algebra of the center of H_ℓ), so the eigenvalues $\nu_1, \ldots, \nu_4$ of $\log(u)$ are contained in a 1-dimensional $\mathbb{Q}$-subspace of an algebraic closure of $\mathbb{Q}_\ell$. This implies in particular that the multiplicative group M generated by $\exp(\nu_1), \ldots, \exp(\nu_4)$ has rank at most 1. Note that $\exp(\nu_1), \ldots, \exp(\nu_4)$ are the eigenvalues of the linear operator u acting on V_ℓ, but since u acts diagonally through the four embeddings of E in $\overline{\mathbb{Q}}$ we can identify them to the Galois conjugates of u. Using that no nontrivial root of unity can be written as a product of eigenvalues of λ we see that M is free of rank 1. Finally, note that u is not stable under complex conjugation, since $\log u$ belongs to $\mathfrak{u}_E$, so $(\log u)' = -\log u$ and the claim follows exponentiating both members.

From Proposition 8.2.6 it then follows that $\mathbb{Q}(u)$ is an imaginary quadratic subfield of E. We deduce $E = E_0 \cdot \mathbb{Q}(u)$, so E is Galois over $\mathbb{Q}$ (being the compositum of two Galois extensions). A CM Galois field of degree four over $\mathbb{Q}$ has Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (cf. the proof of Lemma 4.2.4), so it admits exactly three sub-extensions of degree 2, two of which are imaginary. Let k' be the unique imaginary quadratic field contained in E and distinct from k. The norm $N_{E/k'}$ must send u to a unit in $\mathcal{O}_{k'}$, and since the latter (thanks to Dirichlet's Unit Theorem) has unit group of rank zero it follows that $N_{E/k'}(u)$ is a root of unity; on the other hand, it is a product of conjugates of u, so it cannot be a nontrivial root of unity, and it follows that $N_{E/k'}(u) = 1$.

Differentiating and extending scalars we find $\mathfrak{c} \subseteq \mathfrak{su}_{E/k'} \otimes \mathbb{Q}_\ell$, and this must be an equality because both algebras are of dimension 1. We now get a contradiction exactly as in the geometric case.

The above proves that $\mathfrak{c}$ cannot be 1-dimensional, so we must have $\mathfrak{c} = \mathfrak{u}_E$ (resp. $\mathfrak{u}_{E_l}$)

We now turn to studying the semisimple part of $\mathfrak{h}$. The $\mathfrak{h}_{\mathbf{C}}$-module $V_{\mathbf{C}}$ decomposes as a direct sum $V_{\mathbf{C}} \cong V_1 \oplus V_2$, where each V_i is a free module of rank 2 over $E \otimes_{E_0,\rho_i} \mathbf{C} \cong \mathbf{C} \times \mathbf{C}$. Each V_i, in turn, splits as $V_{\sigma_i} \oplus V_{\tau_i}$, where, as usual, for each $\rho \in R$ we have set

$$V_\rho = \{v \in V_{\mathbf{C}} | e \cdot v = \rho(e)v \quad \forall e \in E\}.$$

Let ψ_i be the restriction of ψ to V_i. Observe that for $v_1 \in V_1, v_2 \in V_2$ and every $e \in E_0$ we have

$$\psi(e \cdot v_1, v_2) = \psi(v_1, e' \cdot v_2) = \psi(v_1, e \cdot v_2),$$

since the Rosati involution is the transpose map with respect to ψ, and on the other hand it is the identity on E_0. This implies

$$\rho_1(e)\psi(v_1, v_2) = \rho_2(e)\psi(v_1, v_2)$$

for every e, so (as $\rho_1 \neq \rho_2$) the only possibility is $\psi(v_1, v_2) \equiv 0$, hence V_1, V_2 are orthogonal with respect to ψ. As ψ is non-degenerate on $V_1 \oplus V_2$, this forces the restriction ψ_i of ψ to V_i to be non-degenerate, and the action of $\mathfrak{h}_{\mathbf{C}}$ must preserve both ψ_1 and ψ_2. We can therefore write $\mathfrak{h}_{\mathbf{C}} \subseteq \mathfrak{u}(V_1, \psi_1) \oplus \mathfrak{u}(V_2, \psi_2)$, so that the semisimple part of $\mathfrak{h}_{\mathbf{C}}$ is contained in $\mathfrak{su}(V_1, \psi_1) \oplus \mathfrak{su}(V_2, \psi_2)$. Moreover, we get the following description for the Abelian part $\mathfrak{c}_{\mathbf{C}}$:

$$\left\{ \left(\underbrace{z_1 \cdot \mathrm{id}}_{\text{on } V_{\sigma_1}}, \underbrace{-z_1 \cdot \mathrm{id}}_{\text{on } V_{\tau_1}} \right), \left(\underbrace{z_2 \cdot \mathrm{id}}_{\text{on } V_{\sigma_2}}, \underbrace{-z_2 \cdot \mathrm{id}}_{\text{on } V_{\tau_2}} \right) \right\} \subset \mathfrak{u}(V_1, \psi_1) \oplus \mathfrak{u}(V_2, \psi_2)$$

Since the projection of $\mathfrak{h}^{ss}$ on each factor $\mathfrak{u}(V_i, \psi_i)$ is nonzero (V_{σ_i} is an irreducible $\mathfrak{h}_{\mathbf{C}}$-module) and the algebras $\mathfrak{su}(V_i, \psi_i)$ are simple (semisimple of rank 1), we see that either the inclusion $\mathfrak{h}^{ss}_{\mathbf{C}} \subseteq \mathfrak{su}(V_1, \psi_1) \oplus \mathfrak{su}(V_2, \psi_2)$ is an equality, or $\mathfrak{h}^{ss}_{\mathbf{C}}$ is the graph of an isomorphism between $\mathfrak{su}(V_1, \psi_1)$ and $\mathfrak{su}(V_2, \psi_2)$.

In order to exclude this second case we simply need to exhibit an element of the form $(0, x) \in \mathfrak{h}^{ss}_{\mathbf{C}}$ with $x \neq 0$. This can be done by comparing the Hodge (resp. Hodge-Tate) decomposition of $V_{\mathbf{C}}$ and the one given by the action of $\mathfrak{h}$. Write $V_j^{(-i,-1+i)}$ for $V_j \cap V_{\mathbf{C}}^{(-i,-1+i)}$ (resp. $V_j \cap V_{\mathbf{C}}(i)$). Using the equalities established at the beginning of this Chapter and comparing dimensions (using $(*)$) we get the following equalities:

$$V_{\sigma_1} = V_1^{(-1,0)}, \quad V_{\tau_1} = V_1^{(0,-1)}$$

$$V_{\sigma_2} = V_{\sigma_2}^{(-1,0)} \oplus V_{\sigma_2}^{(0,-1)}, \quad V_{\tau_2} = V_{\tau_2}^{(-1,0)} \oplus V_{\tau_2}^{(0,-1)},$$

where the dimension of the various spaces are given by

$$\dim_{\mathbf{C}} V_1^{(i,-1+i)} = 2, \quad \dim_{\mathbf{C}} V_{\sigma_2}^{(i,-1+i)} = 1, \quad \dim_{\mathbf{C}} V_{\tau_2}^{(i,-1+i)} = 1.$$

For example, the last equalities are obtained by noticing that

$$\dim\left(V_{*_2}^{(-1,0)}\right) = n_{*_2} = 1,$$

so $V_{*_2} \cong V_{*_2}^{(-1,0)} \oplus V_{*_2}^{(0,-1)}$ (where $* \in \{\sigma,\tau\}$); the others are similar.

The finer decomposition of V we need is

$$V \cong \underbrace{V_1^{(-1,0)}}_{V_{\sigma_1}} \oplus \underbrace{V_1^{(0,-1)}}_{V_{\tau_1}} \oplus V_{\sigma_2}^{(-1,0)} \oplus V_{\tau_2}^{(-1,0)} \oplus V_{\sigma_2}^{(0,-1)} \oplus V_{\tau_2}^{(0,-1)}.$$

Let $\mu : \mathbb{G}_{m,\mathbf{C}} \to \operatorname{End}(V_{\mathbf{C}})$ the cocharacter given on points by

$$\mu(z) = z \operatorname{id}_{V_{\mathbf{C}}^{-1,0}} \oplus \operatorname{id}_{V_{\mathbf{C}}^{0,-1}}.$$

Thanks to the alternative characterization of the Mumford-Tate group (cf. 3.1.2 for the geometric case and Sen's Theorem for the ℓ-adic one) we know that μ factors through G (resp. G_ℓ), so $\mu(z) - \frac{z}{2}\operatorname{id}$ (being traceless) is an element of $\mathfrak{h}$. Write

$$\mu(z) - \frac{z}{2}\operatorname{id} = \left(\mu_1(z) - \frac{z}{2}\operatorname{id}, \mu_2(z) - \frac{z}{2}\operatorname{id}\right) \in \mathfrak{u}\left(V_1,\psi_1\right) \oplus \mathfrak{u}\left(V_2,\psi_2\right)$$

and note that our description of $\mathfrak{c}_{\mathbf{C}}$ implies that $\left(\mu_1(z) - \frac{z}{2}\operatorname{id}, 0\right) \in \mathfrak{c}_{\mathbf{C}}$, so $\tilde{h} := \left(0, \mu_2(z) - \frac{z}{2}\operatorname{id}\right)$ belongs to $\mathfrak{h}_{\mathbf{C}}$. In matrix terms and with respect to the above decomposition of V, $\tilde{h}$ acts as

$$\begin{pmatrix} \underline{0}_4 & & & & \\ & z/2 & & & \\ & & z/2 & & \\ & & & -z/2 & \\ & & & & -z/2 \end{pmatrix},$$

so its trace on both V_{σ_2} and V_{τ_2} is zero, hence $\mu_2(z) - \frac{z}{2}\operatorname{id}$ belongs to $\mathfrak{su}(V_2,\psi)$.

It follows that $\tilde{h}$ belongs to the semisimple part of $\mathfrak{h}_{\mathbf{C}}$, thus giving the element we were looking for and showing that $\mathfrak{h}_{\mathbf{C}}^{ss}$ is all of $\mathfrak{su}\left(V_1,\psi_1\right) \oplus \mathfrak{su}\left(V_2,\psi_2\right)$. By rank considerations it then follows $\mathfrak{h} = \mathfrak{u}_E(V,\psi)$. □

8.2.4.3 Type IV(4,1) - the CM case

There are exactly two possibilities:

1. E does not contain an imaginary quadratic field k acting with multiplicities $\{n_\sigma, n_\tau\} = \{2,2\}$. In this case $\mathfrak{h}$ equals $\mathfrak{u}(V/E)$, which is commutative of rank 4, so Theorem 5.1.13 applies and the Hodge (resp. Tate) conjecture is true for all powers of A.

2. E does contain such a k, $\mathfrak{h} = \mathfrak{su}(V/E)$ and $\mathscr{B}^2(A) = \mathscr{D}^2(A) + \mathscr{W}$. Moreover, $\mathscr{B}^2(A)$ is of dimension 8 and $\mathscr{D}^2(A)$ is of dimension 6.

Proof. In this case A admits complex multiplication, so the statements about the Hodge and the ℓ-adic Lie algebras are equivalent (since the Mumford-Tate conjecture holds). We can therefore restrict ourselves to the geometric case.

Ribet's inequality (Theorem 7.1.3) yields $\operatorname{rank}(\mathfrak{h}) \geq \log_2(2g) = 3$; on the other hand, $\operatorname{rank}(\mathfrak{u}(V/E)) = \operatorname{rank} \operatorname{Res}_{E_0/\mathbb{Q}}(\mathfrak{u}(V,E)) = [E_0 : \mathbb{Q}] = 4$, since V is an E-vector space of dimension 1. It follows that the rank of $\mathfrak{h}$ can only be 3 or 4. Clearly, this rank is 4 exactly when $\mathfrak{h}$ is all of $\mathfrak{u}(V/E)$.

Suppose that $\mathfrak{h}$ is of rank 3. Then Key Lemma 2.4.6 shows that there exists an imaginary quadratic field $k \subset E$ such that $\mathfrak{h} = \mathfrak{su}_{E/k}$, so $\mathfrak{h}$ is in particular semisimple and Lemma 8.1.2 shows that $n_\sigma = n_\tau = 2$.

Conversely, suppose there exists an imaginary quadratic field k contained in E that acts on A with multiplicities $n_\sigma = n_\tau = 2$: then $\mathfrak{h} \subseteq \mathfrak{su}_{E/k}$, and by rank comparison the two must coincide.

We have therefore shown that the above are the only two possibilities; we now only need to check that in case (ii) the dimensions of the involved spaces are the ones claimed. Let $\operatorname{Hom}(k, L) = \{\sigma, \tau\}$, where we have chosen $\sigma_1, \ldots, \sigma_4, \tau_1, \ldots, \tau_4$ in such a way that for each i σ_i restricts to σ on k and τ_i restricts to τ. Proposition 8.2.3 says that the character group of H, or equivalently of $SU(E/k)$, is

$$X^*(H) = \frac{\mathbb{Z} \cdot R}{\langle \sigma_j + \overline{\sigma_j} \text{ for } j = 1, \ldots, 4; \sum_{i=1}^4 \sigma_i; \sum_{i=1}^4 \tau_i \rangle};$$

in particular, $\mathfrak{h} \otimes \mathbf{C}$ is naturally identified with

$$\left\{ (z_1, -z_1), (z_2, -z_2), (z_3, -z_3), (z_4, -z_4) | \sum_{i=1}^4 z_i = 0 \right\} \subset E \otimes \mathbf{C} \cong \bigoplus_{\rho \in R} \mathbf{C}_\rho,$$

where $\mathbf{C}_\rho = \{v \in E \otimes \mathbf{C} | e \cdot v = \rho(e) v \ \forall e \in E\}$ is the 1-dimensional subspace on which $\mathfrak{h}$ acts through the character ρ.

This description allows us to easily compute the dimensions of the invariant subspaces in both $\bigwedge^2 V^*_{\mathbf{C}}$ and $\bigwedge^4 V^*_{\mathbf{C}}$, that are precisely the dimension of the spaces of Hodge classes we are interested in. Let

$$Y = \bigwedge^2 V^*_{\mathbf{C}}, \quad W := \bigwedge^4 V^*_{\mathbf{C}}.$$

The character spaces decomposition of Y is $Y \cong \bigoplus_{\rho_1 \neq \rho_2} Y(-\rho_1 - \rho_2)$; the trivial character spaces are given by those pairs (ρ_1, ρ_2) such that $\rho_1 + \rho_2 = 0$ in $X^*(U_E)$, which happens exactly when $\rho_1 = \overline{\rho_2}$. Analogously, we have a character spaces decomposition for W, that we write as

$$W \cong \bigoplus_{\substack{\rho_1,\ldots,\rho_4 \\ \text{all distinct}}} W(-\rho_1 - \rho_2 - \rho_3 - \rho_4).$$

In $\mathscr{D}^1(A) \otimes \mathbf{C} = \mathscr{B}^1(A) \otimes \mathbf{C} = Y^{\mathfrak{h}}$ we then have exactly four copies of the trivial representation, given by $Y(-\sigma_i - \tau_i)$ for i ranging from 1 to 4. Therefore, the image of the canonical map

$$\left(\mathscr{D}^1 \otimes \mathbf{C}\right) \otimes \left(\mathscr{D}^1 \otimes \mathbf{C}\right) \to \mathscr{D}^2 \otimes \mathbf{C} \subset \bigwedge^4 V_{\mathbf{C}}^* = W$$

is $\bigoplus_{i \neq j} W(-\sigma_i - \tau_i - \sigma_j - \tau_j)$, of dimension 6.

On the other hand, the space of Hodge classes has dimension 8, because a character space $W(-\rho_1 - \rho_2 - \rho_3 - \rho_4)$ is trivial if and only if $\rho_1 + \ldots + \rho_4 = 0$, which happens if (up to renumbering the ρ_i's) $\rho_1 = \overline{\rho_2}$ and $\rho_3 = \overline{\rho_4}$ (6 possibilities), or if the set $\{\rho_1, \ldots, \rho_4\}$ equals one of the sets $\{\sigma_1, \ldots, \sigma_4\}, \{\tau_1, \ldots, \tau_4\}$ (2 possibilities). We therefore have the decomposition

$$\mathscr{B}^2(A) \otimes \mathbf{C} \cong \mathscr{D}^2(A) \otimes \mathbf{C} \oplus \left(\underbrace{W(-\sigma_1 - \sigma_2 - \sigma_3 - \sigma_4)}_{W_\sigma} \oplus \underbrace{W(-\tau_1 - \tau_2 - \tau_3 - \tau_4)}_{W_\tau} \right),$$

and all we need to show is that $W_\sigma \oplus W_\tau \subseteq \mathscr{W}(A)$. But this is easy: write $V_{\mathbf{C}} \cong V_{\mathbf{C},\sigma} \oplus V_{\mathbf{C},\tau}$ as $k \otimes \mathbf{C}$-modules and note that (since taking duals and exterior powers commute)

$$W_\sigma \cong \left(\bigwedge^4 V_{\mathbf{C},\sigma} \otimes \bigwedge^0 V_{\mathbf{C},\tau} \right)^*, \quad W_\tau \cong \left(\bigwedge^0 V_{\mathbf{C},\sigma} \otimes \bigwedge^4 V_{\mathbf{C},\tau} \right)^*,$$

so $\mathscr{W}(A) \otimes \mathbf{C}$ contains

$$\bigwedge_{k \otimes \mathbf{C}} V_{\mathbf{C}}^* \cong \left(\bigwedge^4 V_{\mathbf{C},\sigma} \otimes \bigwedge^0 V_{\mathbf{C},\tau} \right)^* \oplus \left(\bigwedge^0 V_{\mathbf{C},\sigma} \otimes \bigwedge^4 V_{\mathbf{C},\tau} \right)^* \cong W_\sigma \oplus W_\tau,$$

which completes the proof. □

Thanks

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